LATTICE-THEORETIC CHARACTERIZATION OF SOME STRUCTURAL PROPERTIES OF LINEAR MULTIVARIABLE DYNAMICAL CONTROL SYSTEMS

A THESIS

Presented to

The Faculty of the Division of Graduate

Studies and Research

By

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In Partial Fulfillment

of the Requirements for the Degree

Master of Science in Operations Research

Georgia Institute of Technology

March, 1973

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OF LINEAR MULTIVARIABLE DYNAMICAL CONTROL SYSTEMS

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Date approved by Chairman: Mar. 1, 1974

ACKNOWLEDGMENTS

I wish to express my sincere appreciation to Dr. P. Zunde, my thesis advisor, for his continual encouragement and expert guidance throughout the course of this research. His unreserved help was absolutely indispensable for the successful completion of this thesis.

Special thanks are also due to Dr. D. E. Fyffe and Dr. M. S. Bazaraa who served on the reading committee and made valuable suggestions.

I would like to thank Dr. Catherine Aust of the School of Mathematics for her helpful comments on the first three chapters of my thesis.

Finally, I want to thank Mrs. Betty Sims for her conscientious and excellent job of typing.

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GLOSSARY OF SELECTED SYMBOLS

$\langle P, \circ, * \rangle$	A poset P with binary operations $\circ$ and $*$.
$\cap$	The "meet" operation in a poset; the intersection operation in a set.
$\cup$	The "join" operation in a poset; the union operation in a set.
R^k	The k-dimensional Euclidean space.
V	A finite-dimensional linear vector space over the field R of real numbers.
$L(V)$	The lattice formed by the set of all linear subspaces of V.
$L_A^*(V)$	The lattice of A-invariant subspaces of V.
$X^\perp$	The orthogonal complement of the subspace $X \subseteq V$.
$\bar{L}_A^*(V)$	The lattice formed by the orthogonal complements of the elements of $L_A^*(V)$.
X^∇	The principal ideal generated by the element X of a lattice.
X^Δ	The principal dual ideal (filter) generated by the element X of a lattice.
$R(B)$	The range space of a linear operator represented by the matrix B.
$N(B)$	The null space of a linear operator represented by the matrix B.
$K(B)$, $\bar{K}(B)$	Special-structure lattices of invariant subspaces of V.

SUMMARY

The primary objective of the research reported in this thesis is to investigate the possibility of applying lattice-theoretic concepts to the characterization of some structural properties of linear time-invariant multivariable dynamical control systems.

Based on the known fact that the set of all subspaces of a finite-dimensional linear vector space V over an arbitrary field forms a complete complemented atomic modular lattice $L(V)$ with the binary operations of intersection $\cap$: $L(V) \times L(V) \to L(V)$ and summation $+$: $L(V) \times L(V) \to L(V)$, various structures of the lattices of invariant subspaces of V relative to different types of linear operators are investigated and described. Necessary and sufficient conditions for the finiteness of these lattices are given and existence of linear operators which generate desired structures of invariant subspaces is discussed. Properties of the lattice of invariant subspaces are then related to state controllability, output controllability, observability, state uncontrollability and output uncontrollability properties of linear time-invariant dynamical systems. In particular, the set of input matrices for which the system is completely controllable and the set of output matrices for which the system is completely observable are characterized in terms of the sublattices of $L(V)$.

A lattice-theoretic representation of Kalman's state space canonical decomposition scheme is given and some structures of the associated lattice diagram are discussed.

Some additional lattices with special structures that contain the lattices of invariant subspaces as their sublattices are introduced. Using these lattices, generalized controllability and observability subspaces of the state and output spaces of the dynamical system under consideration are characterized.

Finally, application of the properties of these lattices to perfect output controllability, functional output controllability, unknown-input state observability, functional input observability, noninteracting control, decoupling, and other feedback compensation problems related to the dynamical system of interest are demonstrated.

CHAPTER I

INTRODUCTION

1.1 Structural Properties of Linear Dynamical Systems: Literature Survey

In this thesis the bulk of our research effort will be devoted to a lattice-theoretic investigation of some important structural properties of linear time-invariant multivariable dynamical control systems whose evolution is governed by vector equations of the type

$$\begin{aligned} \dot{\underline{x}}(t) &= A\underline{x}(t) + B\underline{u}(t) \\ \underline{y}(t) &= C\underline{x}(t) + D\underline{u}(t), \end{aligned} \tag{S}$$

with the matrices A, B, C, and D consisting only of constant real elements.

By a structural property of a dynamical system we mean a property which depends primarily on the intrinsic structure but not on the inputs and outputs of the system. The important structural properties of interest for our work will be *state controllability*, *output controllability*, *perfect output controllability*, *observability*, and *unknown-input state observability*.

Roughly speaking, *state controllability* means that it is possible to transfer the system from an initial state $\underline{x}_0$ at time t_0 to any final state $\underline{x}_1$ at some later finite time t_1 by the use of an admissible input. Output controllability has a similar meaning with respect to the output of the system. The paramount importance of the theoretical as well as practical aspects of the concept of controllability in the rapidly expanding discipline of control systems theory is well established and needs no further elaboration. There is an abundance of literature on the subject of controllability of linear dynamical systems. Zunde [72], along with a comprehensive list of pertinent references on controllability, presents the chronological development of this concept.

Perfect output controllability is defined as the capability of reproducing, by the output of a dynamical system subject to control actions, any member of a given class of trajectories. The concept of perfect output controllability which was first introduced by Basile and Marro [6,9] is essentially analogous to the concept of functional reproducibility previously investigated by Brockett and Mesarovič [16], Rosenbrock [58] and others. Briefly described, reproducibility refers to the ability of a system to achieve, with its outputs, something which is desired of it. For example, functional reproducibility refers to the capabilities of a system with respect to the generation of specified time functions; asymptotic reproducibility refers to the possibility of approaching a desired behavior with increasing time; pointwise reproducibility refers to the possibility of achieving a desired value of the output at some one point in time and is the same as controllability in common terminology. However, the state space approach adopted by Basile and Marro [9] for the purpose of investigating the concept of perfect output controllability, as well as the results and computational aspects, are somewhat different.

The concept of *observability* is dual to that of controllability. Generally speaking, observability means that it is possible to estimate the initial state of the system from a record of the output. Using the Duality Theorem of Kalman [40], it is possible to directly obtain most of the results concerning observability from state controllability. In essence, the Duality Theorem says that a system is completely observable if and only if the adjoint system is completely state controllable.

Unknown input observability has a similar meaning except that in this case, as the name implies, some or all of the system input functions are completely unknown. In practice, many cases occur in which some of the input variables are inaccessible, so that we can conveniently distinguish the inputs into two classes; control inputs and disturbances. When the system equations are known, it may be possible, even in the presence of disturbances, to deduce the state trajectory from the knowledge of control inputs and system outputs in a finite time interval. This concept, first introduced by Basile and Marro [7,10], in terms of the characterization of particular subspaces of the state space of the system, is in fact a generalization of the ordinary observability criterion in which all of the input functions are known. Guidorzi [32] has investigated some relationships between the unknown input observability subspaces and the characteristic matrices of certain classes of linear time-invariant dynamical systems. He has shown that well-known and widely-used canonical representations can be effectively used for the determination of these subspaces.

With each one of the above structural properties there is inherently associated a subspace of the state or output space of the dynamical system, namely, state controllability subspace, output controllability subspace, perfect output controllability subspace, observability subspace, and unknown-input state observability subspace. The geometric properties induced by certain linear operators on these subspaces can be effectively utilized for the study of various analysis and synthesis problems in the theory and application of linear dynamical systems. It is this geometric framework of the state space approach, in sharp contrast with the classical techniques, that makes both definitions and results intuitively clear.

The concept of invariance of a subspace under a linear operator is strongly connected with controllability and observability of linear dynamical systems. In fact, it is well known [66] that the controllability and observability subspaces of the system (S) are invariant subspaces under the linear operators A and A^T, respectively. Basile and Marro [6] and also independently Wonham and Morse [68] have generalized the concept of invariance of subspaces which has proved immensely useful in the study of many analysis and synthesis problems of linear multivariable systems such as disturbance localization, pole assignment, feedback compensation, decoupling, etc. It must be pointed out that this basically geometric theory of invariant linear subspaces is effectively limited to time-invariant linear dynamical systems. Recently, Morse and Silverman [55] have introduced the concept of a controllability module for the investigation of certain algebraic properties of a broader class of physically significant processes; namely, time-varying linear systems. Using this further generalization of the notion of controllability subspaces, they employ a purely algebraic method and rederive Brunovsky's earlier results on time-varying systems [19]. Their results essentially depend on a theorem due to Doležal [25] which characterizes the range and null spaces of constant rank time-varying matrices.

Throughout the above discussion, attention was focused primarily on the identification of certain subspaces of the state or output space of the dynamical system. Some of the subspaces of the state space and their intrinsic properties have been used by some authors for the investigation of the underlying canonical structures of the state space of linear dynamical systems.

Kalman [40,41] showed that it is always possible to decompose the state space of a linear dynamical system into the direct sum of four invariant linear subspaces and gave the corresponding canonical form of the system equations. A similar result was obtained independently by Gilbert [29] for the linear time-invariant dynamical system (S), with the matrix A having distinct eigenvalues. Later Weiss and Kalman [66] further refined and extended Kalman's original statements and proofs of the decomposition theorems. Weiss [67] considers this problem once more, but formulates some restrictive hypotheses that are essentially equivalent to assuming the dimensions of the subspaces into which the state space is decomposed to be invariant in time. For this case he supplies both the proof of the decomposition theorem and the algorithms for the determination of the canonical forms.

Youla [70] approaches this problem in a different manner. He starts from the requirement of constructing all the realizations of a given realizable weighting pattern, and subsequently shows how realizations associated with any factorization of a given realizable weighting pattern can be divided into four subsystems in parallel, of which one corresponds to the minimal realization. Analyzing the properties of controllability and observability of these subsystems, he then interprets the division under consideration as a version of Kalman's canonical decomposition. However, it must be pointed out that the structural properties considered in the latter are essentially different from those implicitly present in Youla's decomposition (cf. [24]).

In a recent paper, D'Alessandro et al. [24] present a new approach to the theory of canonical decomposition of linear dynamical systems. These authors introduce a new structural property, called *influenceability*, which together with the unobservability property, which they call *invisibility*, constitute the basis for their method. They prove, without any limiting hypotheses, the existence of a canonical decomposition of the state space into subspaces of constant dimensions, the existence of the canonical form of the system equations, and the uniqueness of this form within an equivalence class. This structure theory subsumes all the known results and, in particular, yields a constant decomposition for time-invariant systems.

1.2 Statement and Scope of the Problem

It is known that the set of all subspaces of a finite-dimensional linear vector space V over an arbitrary scalar field F forms a complete complemented atomic modular lattice $L(V)$ with the binary operations of intersection $\cap$ and summation $+$. The set of all invariant subspaces of the linear vector space V under a linear operator A: $V \to V$ is a sublattice of $L(V)$ which is modular and atomic but, in general, not complemented.

In this research various structures of the lattices of invariant subspaces relative to different types of linear operators will be investigated and described. Necessary and sufficient conditions for the finiteness of these lattices will be given and existence of linear operators which generate desired structures of invariant subspaces will be discussed. Properties of the lattices of invariant subspaces will then be related to state controllability, output controllability and observability properties of linear time-invariant dynamical systems of the form

$$\begin{aligned} \dot{\underline{x}}(t) &= A\underline{x}(t) + B\underline{u}(t) \\ \underline{y}(t) &= C\underline{x}(t) + D\underline{u}(t). \end{aligned} \qquad \text{(S)}$$

In particular, the set of input matrices $\{B\}$ for which the system (S) is completely controllable and the set of output matrices $\{C\}$ for which the system (S) is completely observable will be characterized in terms of the sublattices of the lattice $L(V)$. We will also introduce some special-structure lattices that will contain the lattices of invariant subspaces as their sublattices. Using the properties of these lattices, we will characterize the generalized controllability and observability subspaces of the state and output spaces of the system (S).

Finally, we will demonstrate the applicability of the properties of these lattices to perfect output controllability, unknown-input state observability, feedback compensation, and noninteraction control problems related to the dynamical system (S).

1.3 Relevance of the Study

Our radically different approach which is basically algebraic in nature will provide an alternative way to characterize certain structural properties of linear time-invariant dynamical systems. Using our lattice-theoretic method, we can and will demonstrate that most of the existing results and concepts, in the framework of lattice interpretations, become intuitively transparent, arguments and proofs can be presented in a more compact and straightforward form, and generalizations of certain concepts become more self-suggestive. Because of the extensively developed theory of lattices as a distinct branch of abstract algebra and widely explored rich geometric structures and properties of subspaces of finite- and infinite-dimensional linear vector spaces, our approach seems to have the potential for further extensions in the investigation of the state space characteristics of finite- and infinite-dimensional linear dynamical systems. This algebraic approach is obviously in keeping with the current trend in algebraic systems theory which appears to hold the promise for giving rise to the emergence of a unified mathematical theory of dynamical systems.

The pioneering works of Kalman [42,43,44,45] have, all along, been a major continual effort in this direction. Kalman's rigidly algebraic style of exposition, influenced primarily by the development of automata theory, may seem needlessly abstract to readers who were brought up on the Laplace-transform or state-variables type of linear system theory. In due course, however, all the familiar concepts (impulse response function, transfer function, state transition equations, and so on) will make their appearance, frequently in a sharper or more general form. The algebraic approach has many practical advantages: the Lapace-transform and state-variable approaches are merged into a single framework; linear systems over a finite field become a special case of the general theory; new methods are obtained for the effective computation of realizations; and so on [44].

In the early stages of the evolution of the algebraic systems theory, major attempts were also made by Arbib [2,3], Arbib and Zeiger [4] and others to effect a rapprochement between automata theory and control systems theory which further contributed to the unifying potential of this new methodology.

In recent years some papers have appeared along these lines in the literature of control systems theory. Some important works in this category are [14,15,17,18,20,23,34,38,39,46,47,51,52,55,57,60,61,64,69].

The lattice-theoretic approach to the study of linear dynamical systems reported in this thesis, to the best knowledge of the author, has not appeared anywhere in the literature of dynamical control systems and related disciplines. The research topic was originally suggested by Dr. P. Zunde of the School of Information and Computer Sciences and a set of lecture notes prepared by him constitutes the basis for this research work.

CHAPTER II

LATTICES, LINEAR VECTOR SPACES, AND LINEAR OPERATORS

2.1 Dynamical Systems

In this section we will provide a precise formalization of the concept of a dynamical system in the form of an abstract and comprehensive definition which is adapted from a paper by Weiss and Kalman [66].

Definition 2.1.1. A *dynamical system* is a mathematical structure denoted by the septet $(\Sigma, \mathcal{T}, \Omega, \mathcal{U}, \phi, \mathcal{Y}, \psi)$ where

1. Σ is an abstract space called the state space and $\mathcal{T}$ a set of values of time at which the behavior of the system is defined. $\mathcal{T}$ is an ordered subset of the real numbers, with the usual ordering $>$ (or $<$). If $t_1, t_0 \in \mathcal{T}$, the statement $t_1 > t_0$ (or $t_1 < t_0$) will mean that t_1 is in the future (or in the past) with respect to t_0; equivalently, t_0 is in the past (or in the future) with respect to t_1.

2. Ω and $\mathcal{U}$ are abstract spaces with Ω being the set of all functions of time u: $\mathcal{T} \to \mathcal{U}$ which represent the admissible inputs to the system.

3. For any initial time $t \in \mathcal{T}$, any initial state $\underline{x} \in \Sigma$, and any input $\underline{u} \in \Omega$ defined for $t \geq \tau$ (or $t \leq \tau$), states at other times are determined by a given transition function $\underline{\phi}$: $\Omega \times \mathcal{T} \times \mathcal{T} \times \Sigma \to \Sigma$, which is written as $\phi_{\underline{u}}(t,\tau,\underline{x})$. This function has the following properties:

a. $\phi_{\underline{u}}(\tau;\tau,\underline{x}) = \underline{x}$ for any $\underline{u} \in \Omega$, $\underline{x} \in \Sigma$.

b. $\underline{\phi}_u(t;\tau,\underline{x})$ is defined only when $t \geq \tau$ (or $t \leq \tau$).

c. $\underline{\phi}_u(t_2;t_0,\underline{x}) = \underline{\phi}_u\big(t_2;t_1,\underline{\phi}_u(t_1;t_0,\underline{x})\big)$ for all $\underline{u} \in \Omega$, all $t_0,t_1,t_2 \in \mathcal{T}$ such that $t_2 \geq t_1 \geq t_0$ (or $t_2 \leq t_1 \leq t_0$), and all $\underline{x} \in \Sigma$.

d. If $\underline{u}_{[\tau,t]}$ denotes the equivalence class of functions in Ω whose values agree with $\underline{u}$ on the set $[\tau,t] \cap \mathcal{T}$, then

$$\underline{\phi}_u(t;\tau,\underline{x}) = \underline{\phi}_{u_{[\tau,t]}}(t;\tau,\underline{x}).$$

4. Every output of the system at time t is given by the value of a real function ψ: $\mathcal{T} \times \Sigma \rightarrow \mathrm{R}$; where ψ belongs to a given class $\mathcal{Y}$.

5. The functions ϕ and ψ are continuous with respect to suitable topologies defined on Σ, $\mathcal{T}$, Ω, $\mathcal{Y}$, and the reals, as well as the induced product topologies.

For the purpose of illustrating the conceptual implications underlying the above definition, we will consider a special class of dynamical systems in which

i. Σ is finite-dimensional--the system is finite-dimensional.

ii. $\mathcal{T} = \mathrm{R}$; ϕ and ψ are smooth real functions of time t--the system is continuous-time.

iii. ψ is linear in $\underline{x}$ and $\underline{\phi}$ is jointly linear in $\underline{x}$ and $\underline{u}$--the system is linear.

iv. $\mathcal{U}$ is m-dimensional, $\mathcal{Y}$ is p-dimensional--the system is multivariable, i.e., it is a multi-input multi-output system.

Under these special assumptions, it can be proved [66] that the transition function of the most general dynamical system which satisfies the above axioms is a solution of the vector differential equation

$$\dot{\underline{x}}(t) = \underline{f}[\underline{x}(t),\underline{u}(t),t]$$
$$\underline{y}(t) = \underline{g}[\underline{x}(t),\underline{u}(t),t], \qquad (2.1)$$

where

$\underline{x}(t)$ is an n-dimensional state vector.

$\underline{u}(t)$ is an m-dimensional input vector.

$\underline{y}(t)$ is a p-dimensional output vector.

$\underline{f}$: $R^n \times R^m \times T \to R^n$ is a continuous function of all its arguments and R^n and R^m are n- and m-dimensional real spaces, respectively.

In particular, if the dynamical system (2.1) is of the form of a system of first-order linear vector differential equations

$$\dot{\underline{x}}(t) = A(t)\underline{x}(t) + B(t)\underline{u}(t)$$
$$\underline{y}(t) = C(t)\underline{x}(t) + D(t)\underline{u}(t), \qquad (2.2)$$

where $\underline{x}(t)$, $\underline{u}(t)$, and $\underline{y}(t)$ are as defined above; A(t), B(t), C(t), and D(t) are real $n \times n$, $n \times m$, $p \times n$, and $p \times m$ matrices, respectively, then, given the initial condition $\underline{x}(t_0) = \underline{x}_0$, the solution of (2.2) is given by

$$\underline{x}(t) = X(t)[X(t_0)]^{-1}\underline{x}_0 + \int_{t_0}^{t} X(t)[X(\tau)]^{-1}B(\tau)\underline{u}(\tau)d\tau, \qquad (2.3)$$

where X(t) is a fundamental matrix solution of the homogeneous equation

$$\frac{d\underline{x}(t)}{dt} = A(t)\underline{x}(t), \qquad (2.4)$$

i.e.,

$$\frac{dX(t)}{dt} = A(t)X(t),$$

$$\det X(t_0) \neq 0,$$

where by a fundamental matrix solution of the system (2.4) is meant a matrix X(t) which has as its columns n linearly independent solutions of this system.

Equation (2.3) can be written in terms of the transition matrix $\Phi(t,\tau)$, which is defined by

$$\Phi(t,\tau) = X(t)[X(\tau)]^{-1}.$$

This definition is equivalent to stating that Φ satisfies the relations

$$\frac{\partial\Phi(t,t_0)}{\partial t} = A(t)\Phi(t,t_0),$$

$$\Phi(t_0,t_0) = I.$$

Then (2.3) becomes

$$\underline{x}(t) = \Phi(t,t_0)\underline{x}_0 + \int_{t_0}^{t} \Phi(t,\tau)B(\tau)\underline{u}(\tau)d\tau \tag{2.5}$$

and $\underline{y}(t)$ is then given explicitly as

$$\underline{y}(t) = C(t)\Phi(t,t_0)\underline{x}_0 + \int_{t_0}^{t} C(t)\Phi(t,\tau)B(\tau)\underline{u}(\tau)d\tau + D(t)\underline{u}(t).$$

It is easy to verify that the transition function ϕ defined by

$$\phi_{\underline{u}}(t;t_0,\underline{x}_0) = \underline{x}(t),$$

where $\underline{x}(t)$ is given by (2.5), satisfies axiom 3. The remaining axioms are also easy to verify.

2.2 Structural Properties of Linear Dynamical Systems

Here we will very briefly present some relevant definitions and concepts concerning linear time-invariant dynamical systems. Since detailed treatments of these basic concepts are available in most standard texts in linear systems theory such as [13,21,62 and 71], we will keep our discussion to a bare minimum.

Consider the linear time-invariant dynamical system

$$\dot{\underline{x}}(t) = A\underline{x}(t) + B\underline{u}(t)$$
$$\underline{y}(t) = C\underline{x}(t) + D\underline{u}(t), \qquad (S)$$

where $\underline{x}(t) \in R^n$ is an n-component state vector, $\underline{u}(t) \in R^m$ is an m-component input vector, and $\underline{y}(t) \in R^p$ is a p-component output vector; and R^k denotes a k-dimensional Euclidean space. The matrices A, B, C, and D of order $n \times n$, $n \times m$, $p \times n$, and $p \times m$, respectively, are constant real matrices over the field R of real numbers. If B is an arbitrary $n \times m$ constant real matrix, then

$$\mathcal{R}(B) = \{\underline{x}: \underline{x} = B\underline{u} \text{ for some } \underline{u} \in R^m\}$$

is the range of the matrix (operator) B.

The state $\underline{x}(t_0) \in R^n$ of a dynamical system is said to be *controllable* if and only if for some finite time $t_1 > t_0$ there exists an admissible input $\underline{u}_{[t_0,t_1)}$ that will transfer the initial state $\underline{x}(t_0)$ to any arbitrary state $\underline{x}(t_1) \in R^n$. If the state $\underline{x}(t_0)$ of a linear time-invariant system (S) is controllable at some time $t_0 \in \mathcal{T}$, then it is controllable for all $t \in \mathcal{T}$. The set $\mathcal{X}_c$ of all controllable states forms a linear subspace and is called the *controllable subspace* of the state space of the system (S). If $\mathcal{X}_c = R^n$, then the system is said to be *completely controllable;* in other words, the system (S) is completely controllable if every state of the system is controllable. *Controllable outputs, controllable output subspaces,* and *completely output* *controllable* systems are defined in an analogous manner relative to the output space of the system (S).

The system (S) is said to be *k-input controllable,* in the state or output space, if there exists an input matrix B such that the system is completely controllable for some admissible input $\underline{u}(t) \in R^k$. If the system is k-input controllable but not (k-1)-input controllable, then the k-input system is said to be *minimal-input controllable.*

For the system (S), the following statements are equivalent.

1. The system is controllable.
2. The $n \times nm$ matrix $[B \mid AB \mid A^2B \mid \cdots \mid A^{n-1}B]$ has rank n.
3. The rows of the matrix $(sI-A)^{-1}B$ are linearly independent over the field R of real numbers.

The concept of *observability* of a linear dynamical system is closely related to that of state controllability. This relationship is formalized by the Duality Theorem of Kalman [40]. This criterion simply says that the system (S) is controllable (*observable*) if and only if the conjugate system

$$\dot{\underline{\tilde{x}}}(t) = A^*\underline{\tilde{x}}(t) + B^*\underline{\tilde{u}}(t)$$

$$\underline{\tilde{y}}(t) = C^*\underline{\tilde{x}}(t) + D^*\underline{\tilde{u}}(t),$$

where * indicates the conjugate transpose of the unstarred matrix, is observable (controllable). Based on this idea of duality, we can develop the observability criteria from those of state controllability in a direct manner.

The state $\underline{x}(t_0) \in R^n$ of the system (S) is said to be *observable* if there exists a finite time $t_1 > t_0$, $t_1, t_0 \in T$, such that the knowledge of the input $\underline{u}(t)$ and the output $\underline{y}(t)$ over the time interval $[t_0, t_1]$ suffices to determine the state $\underline{x}(t_0) \in R^n$. The set X_0 of all observable states forms a linear subspace and is called the *observable subspace* of the state space of the system (S). If $X_0 = R^n$, then the system is said to be *completely observable;* in other words, the system (S) is completely observable if every state of the system is observable.

For the system (S), the following statements are equivalent.

1. The system is observable.
2. The n × np matrix $[C^T \mid A^TC^T \mid (A^T)^2C^T \mid \cdots \mid (A^T)^{n-1}C^T]$ has rank n.
3. The columns of the matrix $C(sI-A)^{-1}$ are linearly independent over the field R of real numbers.

2.3 Lattices of Linear Subspaces of a Finite-Dimensional Linear Vector Space

Let V be a finite-dimensional linear vector space over the field R of real numbers with the standard inner product $\langle \underline{x}, \underline{y} \rangle = \sum_{i=1}^{n} x_i y_i$, $\underline{x}, \underline{y} \in V$, and denote by $L(V)$ the set of all linear subspaces of V, i.e. $L(V) = \{X: X \subseteq V\}$. It is clear that $L(V)$ is a partially ordered set under the set-inclusion relation $\subset$. Let us further define the operations $\cap$: $L(V) \times L(V) \to L(V)$, called "intersection," and $+$: $L(V) \times L(V) \to L(V)$, called "summation," which satisfy the *idempotency, commutativity, associativity,* and *absorption* laws. In order to describe these laws let $\circ$ denote an arbitrary binary operation $(X,Y) \mapsto X \circ Y$ on a set S. This operation is said to be *idempotent:* when $X \circ X = X$,

commutative: when $X \circ Y = Y \circ X$,

associative: when $X \circ (Y \circ Z) = (X \circ Y) \circ Z$,

in all three cases for all elements $X,Y,Z \in S$. Two arbitrary binary operations $\circ$ and $\square$ on a set S are said to satisfy the *absorption law* if and only if $X \circ (X \square Y) = X \square (X \circ Y) = X$ for all elements $X,Y \in S$.

From the properties of subspaces of V it follows that if $X_i \in L(V)$, i = 1,2,...,k, then the intersection

$$\bigcap_{i=1}^{k} X_i = \{\underline{x} \in V: \underline{x} \in X_i, \ i=1,2,\ldots,k\}$$

is the largest subspace of V contained in X_i, i=1,2,...,k, and the linear sum

$$\sum_{i=1}^{k} X_i = \{\sum_{i=1}^{k} \underline{x}_i \in V: \underline{x}_i \in X_i, \ i=1,2,\ldots,k\}$$

is the smallest subspace of V containing X_i, i = 1,2,...,k.

It is a well known fact that $L(V)$ is a complete complemented atomic modular lattice, where the lattice operations $\cap$ and $\cup$ are just $\cap$ and +, respectively. Because of the central role that the lattice $L(V)$ will play in the subsequent theoretical developments, it will be worthwhile to elaborate on some of the relevant underlying properties of $L(V)$ in more detail and formalize them in the form of theorems.

Theorem 2.3.1. $\langle L(V),\cap,+\rangle$ is a lattice with V as its greatest element and {0}, the null space, as its least element.

Proof. We need to show that $\langle L(V),\cap,+\rangle$ satisfies conditions L1-L4 of Theorem 2 of the Appendix. Conditions L1-L3 clearly follow from the properties of subspaces of V. To see that L4 also holds, let $M,N,P \in L(V)$. We want to show that

$$M \cap (M+N) = M + (M\cap N) = M.$$

Let $\underline{x} \in M \cap (M+N)$. Then $\underline{x} \in M$ and $\underline{x} = \underline{x}_1 + \underline{x}_2 \in M + N$. This implies that $\underline{x}_1 + \underline{x}_2 \in M$ and $\underline{x}_1 \in M$ and $\underline{x}_2 \in N$. Since M is a subspace of V, we have $\underline{x}_2 \in M$ and $\underline{x}_1 \in M$ and $\underline{x}_2 \in N$. Thus $\underline{x}_1 \in M$ and $\underline{x}_2 \in M \cap N$. Therefore $\underline{x}_1 + \underline{x}_2 \in M + (M\cap N)$. Hence $M \cap (M+N) \subseteq M + (M\cap N)$. The reverse inclusion can be shown in a similar manner. Thus $M \cap (M+N) = M + (M\cap N)$. To complete the proof of the first part of the theorem we need to show that $M + (M\cap N) = M$. But this is clear from the fact that $M \cap N \subseteq M$.

The second statement of the theorem is obvious since for any subspace M of V we have $\{0\} \subseteq M \subseteq V$.

For simplicity of notation we will henceforth denote the least element $\{0\}$ of the lattice $L(V)$ by O.

<u>Theorem 2.3.2.</u> $\langle L(V),\cap,+\rangle$ is a complete modular lattice.

Proof. It is immediately clear that $L(V)$ is a complete lattice since for any subset S of $L(V)$, the join $\sum_i M_i$, $M_i \in S$ and the meet $\bigcap_i M_i$, $M_i \in S$ exist in S. To prove modularity, we need to show that if $M \subseteq P$, then

$$P \cap (M+N) = (P\cap M) + (P\cap N) = M + (P\cap N) \quad \text{for all } M,N,P \in L(V). \tag{2.6}$$

We note first that $P \cap M \subseteq P \cap (M+N)$ and $P \cap N \subseteq P \cap (M+N)$. Hence

$$(P\cap M) + (P\cap N) \subseteq P \cap (M+N). \tag{2.7}$$

To show the reverse inclusion, let $\underline{x} \in P \cap (M+N)$. Then $\underline{x} \in P$ and $\underline{x} = \underline{x}_1 + \underline{x}_2 \in M + N$. This implies that $\underline{x}_1 + \underline{x}_2 \in P$ and $\underline{x}_1 \in M$ and $\underline{x}_2 \in N$. Since P is a subspace it follows that $\underline{x}_1 \in P$ and $\underline{x}_2 \in P$ and $\underline{x}_1 \in M$ and $\underline{x}_2 \in N$. Thus $\underline{x}_1 \in M$ and $\underline{x}_2 \in P \cap N$. Hence $\underline{x}_1 + \underline{x}_2 \in M + (P\cap N)$. Therefore

$$P \cap (M+N) \subseteq M + (P\cap N) = (P\cap M) + (P\cap N). \tag{2.8}$$

Now (2.6) follows from (2.7) and (2.8).

It is important to note that $L(V)$ is not a distributive lattice, because although in any lattice we have

$$\begin{gathered} \bigcap_{i,j} (M_i+N_j) \subseteq (\bigcap_i M_i) + (\bigcap_j N_j) \\ \sum_{i,j} (M_i\cap N_j) \subset (\sum_i M_i) \cap (\sum_j N_j), \end{gathered} \tag{2.9}$$

equality does not always hold in (2.9). For example, let $\underline{x}_1$ and $\underline{x}_2$ be independent vectors and let $M = \text{span}\{\underline{x}_1\}$, $N = \text{span}\{\underline{x}_1+\underline{x}_2\}$, and $P = \text{span}\{\underline{x}_2\}$, where $M,N,P \in L(V)$. Then $M + N = \text{span}\{\underline{x}_1+\underline{x}_2\}$ so that

$P \cap (M+N) = M$. On the other hand, $P \cap M = 0$ and $P \cap N = 0$ so that $(P \cap M) + (P \cap N) = 0$. Hence $P \cap (M+N) \neq (P \cap M) + (P \cap N)$.

Theorem 2.3.3. $\langle L(V), \cap, + \rangle$ is a complemented lattice.

Proof. We know that $L(V)$ has the null space 0 as its least and V as its greatest element. We have to show that an arbitrary subspace $M \subseteq V$ has a complement, i.e., for all $M \in L(V)$ there exists an $N \in L(V)$ such that $M + N = V$ and $M \cap N = 0$. Let $[\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_m]$ be a basis of M. These vectors are linearly independent and can therefore be extended to a basis $[\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_m, \underline{x}_{m+1}, \ldots, \underline{x}_n]$ of V. Let N be the space spanned by the vectors $\underline{x}_{m+1}, \underline{x}_{m+2}, \ldots, \underline{x}_n$. Then $M + N = \text{span}\{\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_m, \underline{x}_{m+1}, \ldots, \underline{x}_n\} = V$ and $M \cap N = 0$.

Theorem 2.3.4. $\langle L(V), \cap, + \rangle$ is an atomic lattice.

Proof. It is clear that the atoms of $L(V)$ are the one-dimensional subspaces of V, since they cover the zero element 0 of $L(V)$, i.e., they properly contain the null space 0.

Recall that if M is any subspace of the finite-dimensional linear vector space V, then its orthogonal complement $M^{\perp}$, i.e., the set of all vectors in V which are orthogonal to every vector in M, is always a subspace of V. It is easy to show that if $M,N \subseteq V$ and $M \subseteq N$, then $M^{\perp} \supseteq N^{\perp}$, i.e., the operation of complementation reverses the set-inclusion partial ordering.

Theorem 2.3.5. $\langle L(V), \cap, + \rangle$ is an orthocomplemented lattice.

Proof. Let $M, N \in L(V)$. Then $M^{\perp}$ is a complement of M; if $M \subseteq N$ then $M^{\perp} \supseteq N^{\perp}$, and $M^{\perp\perp} = M$. The last equality follows from the fact that any linear subspace M of a finite-dimensional linear vector space V is orthogonally closed, i.e., $M^{\perp\perp} = M$ [35].

2.4 Morphisms of $L(V)$

In this section we will briefly discuss some relevant properties of the morphisms of the lattice $L(V)$. Later we will make use of these properties for the lattice-theoretic characterization of state and output controllability of the linear dynamical system (S).

Let σ be a linear mapping of the lattice $L(V)$ into the lattice $L(W)$, defined as follows: For all $M \in L(V)$ and all $N \in L(W)$

$$\sigma(M) = \{\underline{x}:\ \ \underline{x} \in N,\ \underline{y} \in M \text{ and } A\underline{y} = \underline{x}\},$$

where A is a matrix of appropriate dimensions representing the linear mapping T of the n-dimensional linear vector space V into the subspace W of V. In particular, if W is an m-dimensional subspace, then A is an $m \times n$ matrix. σ defined as above, is said to be a *lattice morphism* induced by the linear mapping T: $V \to W$. If $V = W$, then a (order or lattice) homomorphism σ: $L(V) \to L(W)$ is called an (*order* or *lattice*) *endomorphism*. Finally, a lattice homomorphism σ: $L(V) \to L(W)$ which is both a monomorphism (injection) and an epimorphism (surjection) is called an *isomorphism*; if $L(V)$ is the same lattice as $L(W)$, then σ is called an *automorphism*.

Theorem 2.4.1. Let V and W be n-dimensional and m-dimensional linear vector spaces, respectively. Then the linear mapping $\sigma\colon L(V) \to L(W)$ is a join-homomorphism (+-homomorphism).

Proof. We need to show that for any $M,N \in L(V)$, $\sigma(M+N) = \sigma(M) + \sigma(N)$. Let $\underline{y} \in \sigma(M+N)$. Then there exists an $\underline{x} = \underline{x}_1 + \underline{x}_2 \in M + N$ such that $\underline{y} = \sigma(\underline{x}) = \sigma(\underline{x}_1+\underline{x}_2) = \sigma(\underline{x}_1) + \sigma(\underline{x}_2)$ and since $\underline{y}$ is arbitrary, we obtain

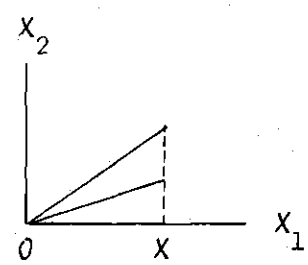
$$\sigma(M+N) \subseteq \sigma(M) + \sigma(N). \tag{2.10}$$

On the other hand, if $\underline{x} = \underline{x}_1 + \underline{x}_2 \in \sigma(M) + \sigma(N)$, then for some $\underline{y} \in M$, $\underline{x}_1 = \ (\underline{y})$ and some $\underline{z} \in N$, $\underline{x}_2 = \sigma(\underline{z})$. Now $\underline{x}_1 + \underline{x}_2 = \sigma(\underline{y}) + \sigma(\underline{z}) = (\underline{y}+\underline{z})$. This implies that

$$\sigma(M) + \sigma(N) \subseteq \sigma(M+N) \tag{2.11}$$

From (2.10) and (2.11) it follows that for all $M,N \in L(V)$, $\sigma(M+N) = \sigma(M) + \sigma(N)$.

Notice that the lattice morphism $\sigma\colon L(V) \to L(W)$ induced by the linear mapping $T\colon V \to W$ is not in general a meet-homomorphism ($\cap$-homomorphism) as the following example shows. Let P be a projection mapping which maps any linear subspace in the plane onto the X_1-axis as illustrated in the adjacent figure. It is clear that



$$P(M \cap N) = PO = O$$

and

$$P(M) \cap P(N) = X \neq O.$$

Hence

$$P(M \cap N) \neq P(M) \cap P(N).$$

Let $\bar{L}(V)$ denote the set of orthogonal complements of all the elements of the lattice $L(V)$. If to each linear subspace M of V we make correspond its orthogonal complement $M^{\perp}$ in V, then from the fact that every linear subspace of a finite-dimensional linear vector space V is orthogonally closed we have a one-to-one correspondence of $L(V)$ with $\bar{L}(V)$ in which the operation of complementation reverses the set-inclusion partial ordering. We summarize this observation in the following theorem.

Theorem 2.4.2. The correspondence which sends each linear subspace M of V to its orthogonal complement $M^{\perp}$ in V is a dual isomorphism of the lattices $L(V)$ and $\bar{L}(V)$. $\bar{L}(V)$ is thus also a complete complemented atomic modular lattice.

This conclusion is essentially equivalent to the orthocomplementation property of the lattice $L(V)$. Since $L(V)$ is an orthocomplemented lattice, for all $M,N \in L(V)$, $M \subseteq N$ implies that $M^{\perp} \supseteq N^{\perp}$ and $M^{\perp\perp} = M$. By these two conditions, the orthocomplementation $M \mapsto M^{\perp}$ is a dual isomorphism of $L(V)$ onto $\bar{L}(V)$. From this fact we have

$$O^{\perp} = V,\ V^{\perp} = O,\ (M+N)^{\perp} = M^{\perp} \cap N^{\perp},$$

and

$$(M \cap N)^{\perp} = M^{\perp} + N^{\perp}, \quad \text{for all } M,N \in L(V).$$

We state this result as a corollary to Theorem 2.4.2.

Corollary 2.4.1. If M_i, i = 1,2,...,k are subspaces of V, then

$$\left(\bigcap_{i=1}^{k} M_i\right)^{\perp} = \sum_{i=1}^{k} M_i^{\perp} \quad \text{and} \quad \left(\sum_{i=1}^{k} M_i\right)^{\perp} = \bigcap_{i=1}^{k} M_i^{\perp}.$$

2.5 Lattices of Invariant Subspaces of a Finite-Dimensional Linear Vector Space

The concept of invariance of linear subspaces of the state and output spaces of the linear dynamical system (S) under certain linear operators is closely related to some important structural properties of the system (S), such as state controllability, state observability, and output controllability. In fact it is known that the controllability subspace, i.e., the set of all controllable states of the system (S) is the smallest A-invariant subspace that contains the range $R(B)$ of the input matrix B and the observability subspace, i.e., the set of all observable states of the system (S) is the greatest A^T-invariant subspace that is contained in the orthogonal complement $(R(B))^{\perp}$ of the subspace $R(B)$ [66].

For the purpose of lattice-theoretic investigation of some structural properties of the system (S), below we will study and formalize in detail the properties of the lattice formed by the set of invariant subspaces of the linear vector space V under certain linear operators.

Let V be a finite-dimensional linear vector space over the field R of real numbers and A the matrix of a linear operator on V. If W is a subspace of V, we say that W is *invariant under* A or A-*invariant* if for each vector $\underline{x} \in W$ the vector $A\underline{x}$ is in W, i.e., $AW \subseteq W$. In particular, if $AW = W$, then W is said to be *point invariant under* A or *point* A-*invariant.*

Consider now the set

$$L_A^*(V) = \{M \in L(V):\ AM \subseteq M\}, \tag{2.12}$$

i.e., the set of all A-invariant subspaces of the linear vector space V. It is clear that $L_A^*(V)$ is a partially ordered set under the set-inclusion relation $\subset$. It will be shown below that $L_A^*(V)$ is a lattice. Brickman and Fillmore [12] have investigated some properties and structures of this lattice. However, our approach to the investigation of the properties and structures of $L_A^*(V)$ and some of our results are completely different.

In the next few theorems we will investigate and formalize the properties of $L_A^*(V)$. First we need a lemma whose simple proof is omitted.

<u>Lemma 2.5.1</u>. If M and N are A-invariant subspaces of V, then $M \cap N$ and $M + N$ are also A-invariant subspaces of V.

Theorem 2.5.1. $L_A^*(V)$ is a sublattice of $L(V)$.

Proof. Let $M,N \in L_A^*(V)$. By Lemma 2.5.1, $M \cap N \in L_A^*(V)$ and $M + N \in L_A^*(V)$ and hence $L_A^*(V)$ is a sublattice of $L(V)$.

Theorem 2.5.2. $L_A^*(V)$ has as its smallest element the zero element 0 and as its greatest element the unit element V of $L(V)$.

Proof. Since $A0 = 0$ and $AV \subseteq V$, 0 and V are elements of $L_A^*(V)$. But $L_A^*(V)$ is a sublattice of $L(V)$, hence 0 is the smallest element and V is the greatest element of $L_A^*(V)$.

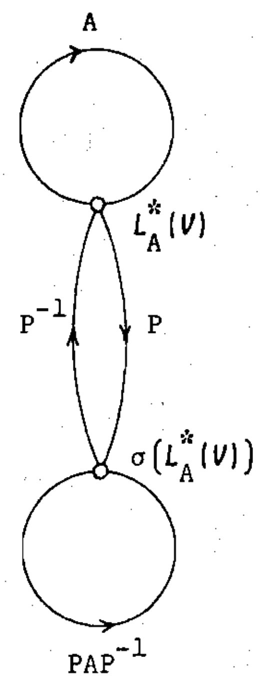
Theorem 2.5.3. The lattice $L_A^*(V)$ is modular and atomic.

Proof. The proof of modularity follows from the fact that every sublattice of a modular lattice is modular [26]. To show atomicity, let $M \in L_A^*(V)$. Since M is an A-invariant subspace of V and therefore contains 0, there exists an interval $[0,M]$ of finite length in $L_A^*(V)$. If there is no $X \in L_A^*(V)$ such that $0 \subsetneq X \subsetneq M$, then M is an atom, and the requirement is satisfied. If there is an $X_1 \in L_A^*(V)$ such that $0 \subsetneq X_1 \subsetneq M$, then either the interval $[0,X_1]$ is simple in which case X_1 is an atom and $X_1 \subsetneq M$, so that M contains an atom, or $[0,X_1]$ is not simple in which case there is an X_2 such that $0 \subsetneq X_2 \subsetneq X_1$. If X_2 is such that the nterval $[0,X_2]$ is simple, then X_2 is an atom and $X_2 \subsetneq X_1 \subsetneq M$. Otherwise we proceed further in this manner. This process has to terminate since the interval $[0,M]$ is of finite length and every interval $[0,X_{i+1}]$ is properly contained in the interval $[0,X_i]$ so that $[0,X_{i+1}]$ is shorter than $[0,X_i]$ for all $X_i \in L_A^*(V)$. Thus continuing with the above constructive procedure, we will eventually find an $X_m \in L_A^*(V)$ such that the interval $[0,X_m]$ is simple and hence X_m is an atom contained in M.

It must be pointed out that although $L(V)$ is a complemented lattice, $L_A^*(V)$ is not necessarily complemented.

Theorem 2.5.4. Let σ: $L_A^*(V) \to L_A^*(V)$ be a lattice endomorphism induced by the operator P: $V \to V$ which is one-to-one but otherwise arbitrary. Then $\sigma\left(L_A^*(V)\right)$ is isomorphic to $L_A^*(V)$ and the lattice $\sigma\left(L_A^*(V)\right)$ is a lattice of invariant subspaces with respect to the operator PAP^{-1}.

Proof. The first statement of the theorem immediately follows from the fact that a lattice morphism is invertible if and only if it is a lattice isomorphism [26] and the second statement of the theorem is obvious from the following mapping diagram.



Corollary 2.5.1. The lattices $L_A^*(V)$ and $L_{PAP^{-1}}^*(V)$, where A and PAP^{-1} are similar matrices of the linear operator T: $V \to V$, are isomorphic and their fixed points are the same.

This corollary is important in investigating some structural properties of the system (S) that are invariant under equivalence transformations, such as controllability and observability.

The set of orthogonal complements of all the elements of the lattice $L_A^*(V)$, denoted by $\bar{L}_A^*(V)$, will play an important role in the lattice-theoretic characterization of the observability properties of the system (S). Here we will investigate and formalize some properties of the set $\bar{L}_A^*(V)$.

Theorem 2.5.5. All elements of $\bar{L}_A^*(V)$ are A^T-invariant subspaces of V.

Proof. Let $M \in L_A^*(V)$, thus $AM \subseteq M$, i.e., for all $X \in M$, $A\underline{x} \in M$. Suppose $\underline{y}$ is an element of the orthogonal complement of M, i.e., $\underline{y} \in M^\perp$. Then $\langle A\underline{x},\underline{y}\rangle = 0$. But $\langle A\underline{x},\underline{y}\rangle = \langle \underline{x},A^T\underline{y}\rangle = 0$. This implies that $A^T\underline{y} \in M^\perp$. Since $\underline{y} \in M^\perp \Rightarrow A^T\underline{y} \in M^\perp$, $M^\perp$ is an A^T-invariant subspace of V.

Lemma 2.5.2. If $M^\perp$ and $N^\perp$ are A^T-invariant subspaces of V, then $M^\perp \cap N^\perp$ and $M^\perp + N^\perp$ are also A^T-invariant subspaces of V.

Theorem 2.5.6. $\bar{L}_A^*(V)$ is a sublattice of $L(V)$.

Proof. Let $M^\perp, N^\perp \in \bar{L}_A^*(V)$. By Lemma 2.5.2, $M^\perp \cap N^\perp \in \bar{L}_A^*(V)$ and $M^\perp \cap N^\perp \in \bar{L}_A^*(V)$. Since $M^\perp \cap N^\perp \in L(V)$ and $M^\perp + N^\perp \in L(V)$, it follows that $\bar{L}_A^*(V)$ is a sublattice of $L(V)$.

Theorem 2.5.7. $L_A^*(V)$ is dually isomorphic to $\bar{L}_A^*(V)$.

Proof. The proof follows from Theorem 2.4.2.

2.6 Lattices of Invariant Subspaces Generated by the Jordan Canonical Form of the Matrix of a Linear Operator

It is well known that the matrix A of every linear operator T on a finite-dimensional linear vector space over the field R of real numbers can be transformed to a block diagonal matrix J(A), called the Jordan canonical form of the matrix A, by an appropriate equivalence transformation [63]. The form of J(A) is shown below.

$$\begin{pmatrix} J_{11}(\lambda_1) & & & & & \\ & J_{21}(\lambda_1) & & & \bigcirc & \\ & & \cdot & & & \\ & & & J_{k1}(\lambda_1) & & \\ & \bigcirc & & & \cdot & \\ & & & & & J_{mq}(\lambda_q) \end{pmatrix} \tag{2.13}$$

For a real eigenvalue λ_i each $J_{ji}(\lambda_i)$, called a Jordan block, is an upper triangular square matrix with λ_i on the diagonal and ones occurring in all places just above the diagonal. A general $J_{ji}(\lambda_i)$ is shown below.

$$\begin{pmatrix} \lambda_i & 1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_i & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & 0 & \cdots & \lambda_i \end{pmatrix} \tag{2.14}$$

on the other hand, for a complex eigenvalue $\lambda_k = \sigma_k \pm i\tau_k$, the Jordan block will have the following quasi-diagonal form [63].

$$\begin{pmatrix} \sigma_k & \tau_k & 1 & 0 & & & & & & & \\ -\tau_k & \sigma_k & 0 & 1 & & & & & & & \\ & & \sigma_k & \tau_k & 1 & 0 & & & & & \\ & & -\tau_k & \sigma_k & 0 & 1 & & & & & \\ & & & & \sigma_k & \tau_k & & & & & \\ & & & & -\tau_k & \sigma_k & & & & & \\ & & & & & & \ddots & & & & \\ & & & & & & & \sigma_k & \tau_k & 1 & 0 \\ & & & & & & & -\tau_k & \sigma_k & 0 & 1 \\ & & & & & & & & & \sigma_k & \tau_k \\ & & & & & & & & & -\tau_k & \sigma_k \end{pmatrix} \tag{2.15}$$

In summary, given any linear operator T on a finite-dimensional linear vector space over the field R of real numbers, there exists a basis in which the matrix A of T is quasi-diagonal, made up of blocks of the form (2.14) and (2.15), where λ_i, i = 1,2,...,m, are the real eigenvalues and $\sigma_k \pm i\tau_k$, k = 1,2,...,s, the complex eigenvalues of A. A method for uniquely determining the sizes of the blocks is described in [63].

From here on, unless otherwise specified, $L_A^*(V)$ is assumed to be a finite lattice.

Based on the vector space theory of the Jordan canonical form and its corresponding lattice-theoretic interpretations, we will formulate, in a straightforward manner, a number of results in the following theorems. Proofs will be provided only when the assertions do not seem to be obvious.

Theorem 2.6.1. Let A_i be a Jordan block in the canonical form of the matrix A, and let $V_i \subseteq V$, i = 1,2,...,p, be the subspace corresponding to that block in the direct sum decomposition of the space V. Let C_i be the set of invariant subspaces of V contained in V_i, i = 1,2,..., p. Then C_i corresponds to a chain $[0,V_i]$ in $L(V)$ of length d_i if A_i is associated with a real eigenvalue and $\frac{d_i}{2}$ if A_i is associated with a complex eigenvalue of the matrix A, where d_i is the dimension of V_i, i = 1,2,...,p.

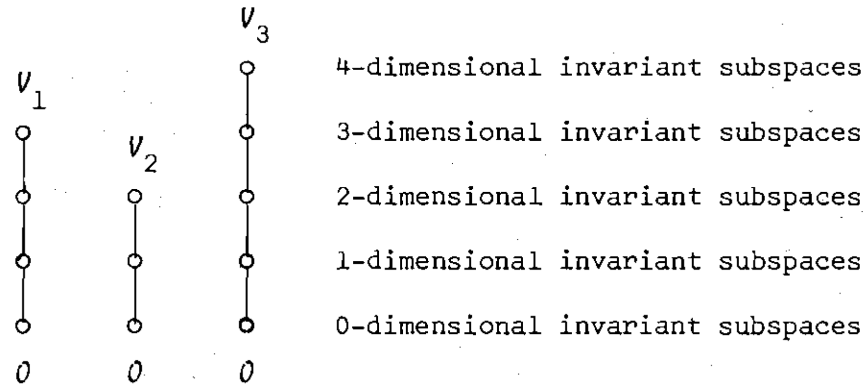
It follows from the above theorem that the number of elements in the chain $[0,V_i]$ associated with the real d_i-dimensional linear subspace V_i is equal to (d_i+1) if the corresponding Jordan block A_i is associated with a real eigenvalue, and is equal to $\left(\frac{d_i}{2} + 1\right)$ if A_i is associated with a complex eigenvalue.

Notice that in the case of a complex eigenvalue d_i will always be an even integer since the complex roots of the characteristic polynomial of A will always appear as complex conjugate pairs, so that $\left(\frac{d_i}{2} + 1\right)$ is an integer [63].

To illustrate the direct connection between the linear vector space direct sum decomposition and the lattice-theoretic formulation in the above theorem, for the case of distinct real eigenvalues, suppose that the invariant subspace V_i in the direct sum decomposition $V = V_1 \oplus V_2 \oplus \cdots \oplus V_p$, has dimension d_i, $i = 1,2,\ldots,p$. This implies that V_i has as its invariant subspaces W_{ij}, $j = 1,2,\ldots,d_i+1$, of dimensions $d_i - k$, $k = 0,1,2,\ldots,d_i$, such that subspaces of dimension $d_i - k$ properly contain subspaces of dimension $d_i - (k+1)$, $k = 0,1,2,\ldots,d_i-1$. In terms of the structure of the lattice $L_A^*(V)$, this is precisely the same thing as saying that W_{ij}, $j = 1,2,\ldots,d_i+1$ form a chain. We will demonstrate this analogy by an example. Suppose that the Jordan canonical form J(A) of a matrix A consists of three blocks A_1, A_2, and A_3, i.e.,

$$J(A) = \begin{bmatrix} A_1 & & \bigcirc \\ & A_2 & \\ \bigcirc & & A_3 \end{bmatrix}$$

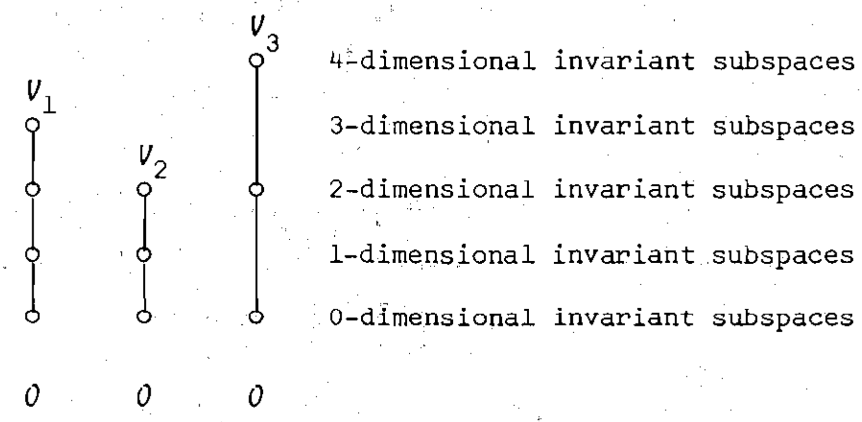
where the submatrices, A_1, A_2, and A_3 are associated with distinct real eigenvalues and are 3 × 3, 2 × 2, and 4 × 4, respectively. Then the chains $[\mathcal{O},\mathcal{V}_i]$ generated by A_i, $i = 1,2,3$ are



For blocks associated with complex distinct eigenvalues of the matrix A and their corresponding invariant subspaces, the above arguments will have to be modified according to the subspace dimensionality property imparted by the existence of the complex eigenvalues. In this case the dimension of a Jordan block is an even integer, say $2d_\rho$, where d_ρ is a positive integer, and the corresponding invariant subspace has subspaces of dimensions $2d_\rho - 2m$, $m = 1,2,\ldots,d_\rho$, such that subspaces of dimension $2d_\rho - 2m$ properly contain subspaces of dimension $2d_\rho - 2(m-1)$, $m = 1,2,\ldots,d_\rho+1$. For example, if

$$J(A) = \begin{bmatrix} A_1 & & \bigcirc \\ & A_2 & \\ \bigcirc & & A_3 \end{bmatrix}$$

where A_1, A_2, and A_3 are 3×3, 2×2, and 4×4, respectively; A_1 and A_2 are associated with distinct real eigenvalues and A_3 with a complex eigenvalue of A. For this case the chains $[\mathcal{O},V_i]$ generated by A_i, $i = 1,2,3$ are shown below.



<u>Theorem 2.6.2.</u> The lattice $L_A^*(V)$ is the direct product of the chains $[\mathcal{O},V_i]$ generated by the Jordan blocks A_i, $i = 1,2,\ldots,p$, of the Jordan canonical form of the matrix A.

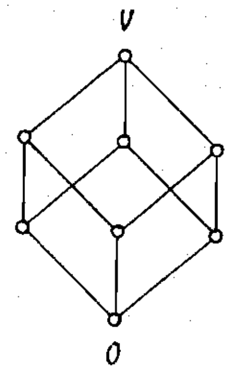
In order to demonstrate the generation scheme for the lattice $L_A^*(V)$, provided by Theorem 2.6.2, we will give several examples of the Jordan canonical forms of different matrices and their associated lattices of invariant subspaces.

Example 1.

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

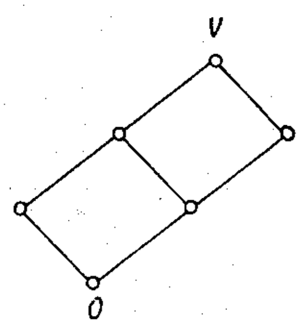
Example 2.

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$



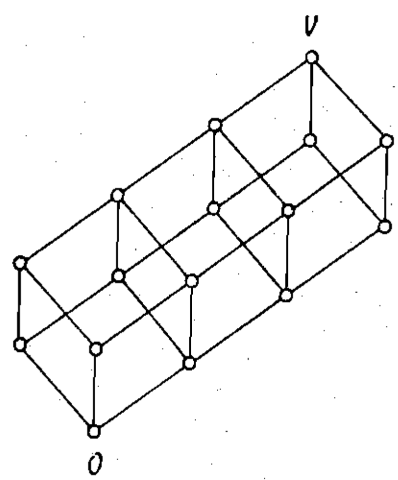
Example 3.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$



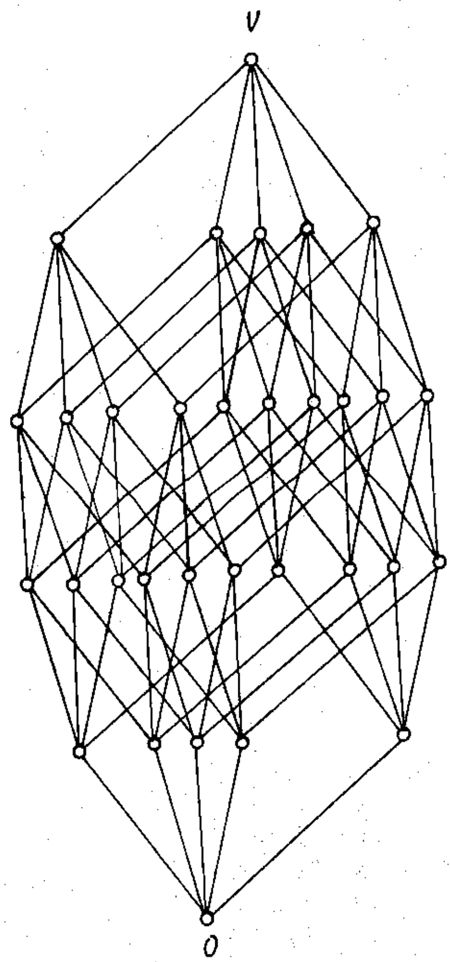
Example 4.

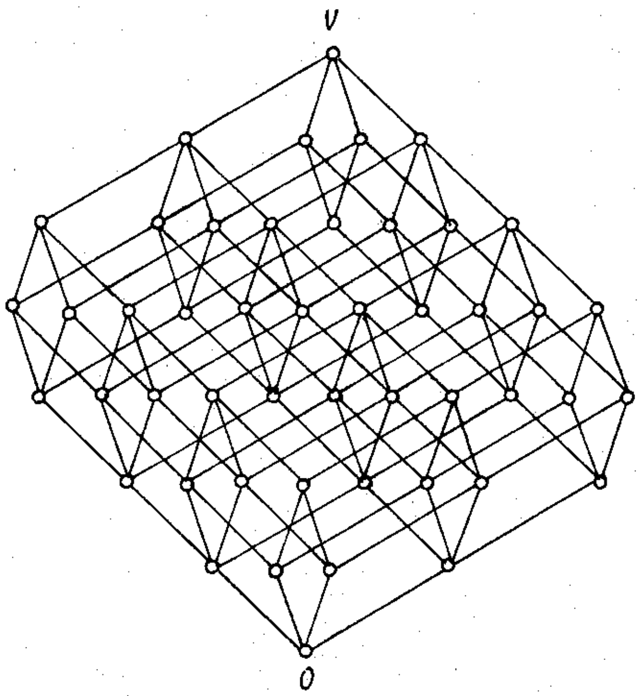
$$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 4 & 1 & 0 \\ 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}$$



Example 5.

$$\begin{pmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 7 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$





Example 6.

$$\begin{pmatrix} 4 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 9 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 7 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 7 \end{pmatrix}$$

It is apparent from the above examples that the complexity of the lattice diagrams increases with the dimension of the matrix A. However, it turns out that for lattices whose generating segments consist only of chains, simple algorithmic generation schemes can be developed. Earlier we indicated that the Jordan canonical form of the matrix of a linear operator with distinct eigenvalues associated with different Jordan blocks generates two types of lattice of invariant subspaces:

1. Lattices whose generating segments consist of chains of unit length; these lattices are called *Boolean lattices* or *Boolean algebras* (Examples 2,5).

2. Lattices whose generating segments consist of chains of varying lengths; these lattices are called *factorization lattices* (Examples 3,4,6).

Formulas for determining the number of elements at different levels of these lattices and simple rules for drawing the lattice diagrams are known to exist for these two types of lattice [26].

Earlier we showed that the lattice $L(V)$ is not, in general, a distributive lattice. However, the sublattice $L_A^*(V)$ of $L(V)$ is distributive as shown in the following theorem.

Theorem 2.6.3. $L_A^*(V)$ is a distributive lattice.

Proof. By Theorem 2.6.1, $[0,V_i]$, i = 1,2,...,p, are chains in the lattice $L(V)$. It is known that a chain is a distributive lattice and the direct product of distributive lattices is a distributive lattice [26].

Theorem 2.6.4. The elements of the chains $[0,V_i]$, i = 1,2,...,p, of the product lattice $L_A^*(V)$ are the irreducible elements of the lattice

$L_A^*(V)$. Consequently, every element of the lattice $L_A^*(V)$ can be expressed as the join of some elements of the chains $[0, V_i]$.

Theorem 2.6.5. The lattice $L_A^*(V)$ is complemented iff the $n \times n$ matrix A has n distinct eigenvalues.

Theorem 2.6.6. The lattice $L_A^*(V)$ is a Boolean algebra iff A has n distinct eigenvalues.

Proof. Suppose that A has n distinct eigenvalues. Then by Theorem 2.6.5 $L_A^*(V)$ is complemented. But by Theorem 2.6.3 $L_A^*(V)$ is a distributive lattice. Therefore $L_A^*(V)$ is a complemented distributive lattice and hence a Boolean algebra. Conversely, if $L_A^*(V)$ is a Boolean algebra and hence complemented, by Theorem 2.6.5, the matrix A has n distinct eigenvalues.

Theorem 2.6.7. $L_A^*(V)$ is a finite lattice iff in the Jordan canonical form of A, each eigenvalue associated with a Jordan block is distinct from an eigenvalue associated with another Jordan block.

Corollary 2.6.1. $L_A^*(V)$ is a finite lattice iff the matrix A has n distinct eigenvalues.

Theorem 2.6.8. $L_A^*(V)$ is an infinite lattice iff in the Jordan canonical form of A two or more blocks are associated with one and the same eigenvalue of A.

CHAPTER III

CONTROLLABILITY AND OBSERVABILITY

3.1 State Controllability

For a state $\underline{x}(t) \in R^n$ of the system (S) to be *controllable*, the necessary and sufficient condition is that $\underline{x}(t)$ be an element of the subspace

$$\mathcal{Q}_c = \text{span}\{B \mid AB \mid A^2B \mid \cdots \mid A^{n-1}B\},$$

that is, $\mathcal{Q}_c$ is the *controllable subspace* of the state space of the system (S) [5]. Clearly $\mathcal{Q}_c$ is a subspace of the linear vector space V and thus is an element of the lattice $L(V)$. Furthermore, since $\mathcal{Q}_c$ is an A-invariant subspace of V, $\mathcal{Q}_c$ is an element of $L_A^*(V)$. We also observe that $\mathcal{Q}_c$ is the smallest A-invariant subspace of V which contains the range of the input matrix B, $R(B) = \{\underline{x}: \underline{x} = B\underline{u}$ for some $\underline{u} \in R^m\}$, i.e.,

$$\mathcal{Q}_c = R(B) + AR(B) + A^2R(B) + \ldots + A^{n-1}R(B)$$

We summarize these observations in the following theorem.

Theorem 3.1.1. The subspace X of the linear vector space V is a controllable subspace of the state space of the system (S) iff

$$X = \inf\{L_A^*(V) \cap [R(B)]^\Delta\}.$$

Corollary 3.1.1. Let $\{M\}$ be the set of all elements of $L_A^*(V)$ which contain $R(B)$. Then the controllable subspace X is $\inf\{M\}$.

This corollary can be equivalently stated as follows.

Corollary 3.1.1'. For some $M \in L_A^*(V)$, M is a controllable subspace iff $\ell[M,R(B)] < \ell[N,R(B)]$ for all $N \in L_A^*(V)$, $M \neq N$.

Corollary 3.1.2. For any input matrix $B \in \{B\}_{n\times m}$ and any $X \in L_A^*(V)$, X is a controllable subspace iff the interval $[X,R(B)]$ does not contain any other elements of $L_A^*(V)$ except the element X.

Let $\dim[R(B)]$ denote the dimension of the range of the input matrix B. Now, $\dim[R(B)]$ is equal to $r(B)$, the rank of the matrix B. If the matrix B is of size $n \times m$, and $m > r(B)$, then $m - r(B)$ of the system inputs are dependent; these are "superfluous" in the sense that they do not affect the output of the system. Consequently, $r(B)$ represents the number of "effective" inputs which can influence the state vector of the system.

We will say that two matrices B_1 and B_2 are input equivalent if $R(B_1) = R(B_2)$ and, unless otherwise stated, any element of the equivalence class thus defined may be taken as a representative of the whole class, namely, the class of all matrices having the same range.

The system (S) is said to be completely state controllable if and only if

$$\mathcal{Q}_c = \text{span}\{B \mid AB \mid A^2B \mid \cdots \mid A^{n-1}B\} = \mathcal{V}.$$

As an immediate consequence of this definition we can characterize the set of all input matrices $\{B\}_{n\times m}$ for which the system (S) is completely state controllable.

Theorem 3.1.2. For any input matrix $B \in \{B\}_{n\times m}$, the system (S) is completely state controllable iff the sublattice $[\mathcal{R}(B),\mathcal{V}]$ of $\mathcal{L}(\mathcal{V})$ has only the unit element in common with the sublattice $\mathcal{L}_A^*(\mathcal{V})$.

Theorem 3.1.3. Let $\mathcal{X} \in \mathcal{L}_A^*(\mathcal{V})$ for some fixed matrix A of the system (S). The set of all input matrices $\{B\}_{n\times m}$ for which $\mathcal{X}$ is a controllable subspace of the state space of the system (S) consists of the matrices whose ranges are elements of the set

$$\mathcal{X}^\nabla = \bigcup_{\substack{\mathcal{M} \subset \mathcal{X} \\ \mathcal{M} \in \mathcal{L}_A^*(\mathcal{V})}} \mathcal{M}^\nabla .$$

Next we will consider the question of existence of an input matrix of the smallest rank for which the system (S) is state controllable. In other words, we wish to determine the condition to be imposed on the matrix B for minimum-input state controllability.

The criterion for minimum-input state controllability is contained in the following theorem.

Theorem 3.1.4. Let X be a controllable subspace of the state space of the system (S) and consider the poset

$$\mathcal{W} = X^{\nabla} - \bigcup_{\substack{M \subset X \\ M \in L_A^*(V)}} M^{\nabla}.$$

Then the minimum number of inputs needed to control the state vector of the system, or equivalently, the rank of the input matrix for which the system is minimum-input state controllable is equal to the dimension of the minimal element in $\mathcal{W}$.

Theorem 3.1.5. Consider the single-input dynamical system

$$\dot{\underline{x}} = A\underline{x} + \underline{b}u. \tag{S'}$$

The system (s') is state controllable for some input vector $\underline{b} \in R^n$ iff the lattice $L_A^*(V)$ is finite.

Proof. Consider the linear transformation $\underline{z} = T\underline{x}$ of the state vector $\underline{x} \in R^n$ which transforms the system (S') into

$$\dot{\underline{z}} = TAT^{-1}\underline{z} + T\underline{b}u, \tag{S''}$$

where TAT^{-1} is the Jordan canonical form of A. It is known that the system (S') is state controllable if and only if:

1. Each eigenvalue associated with a Jordan block is distinct from an eigenvalue associated with another Jordan block.

2. Each element $\tilde{b}_i$ of the vector $\tilde{b} = T\underline{b}$ associated with the bottom row of each Jordan block is nonzero [71].

The theorem now follows from the fact that finiteness of $L_A^*(V)$ is equivalent to condition (1) and obviously we can find an input vector $\underline{b}$ such that $T\underline{b} \neq \underline{0}$ which satisfies condition (2).

If X_c is the controllable subspace of the state space of the system (S), then it is well known that no input can completely control the state vector of the system which is contained in the direct complement $\bar{X}_c$ of the subspace X_c. The subspace $\bar{X}_c$ is called an *uncontrollable subspace* of the state space of the system (S). That is, any subspace $\bar{X}_c$ such that $X_c + \bar{X}_c = V$ and $X_c \cap \bar{X}_c = 0$ qualifies for an uncontrollable subspace. As such uncontrollable subspaces are not unique. However, an uncontrollable subspace can be uniquely specified by choosing it to be the orthogonal complement of X_c. This subspace will be denoted by $X_c^\perp$. Since $X_c^\perp$ is an A^T-invariant subspace, $X_c^\perp \in \bar{L}_A^*(V)$. From the above discussion, the argument preceding Theorem 3.1.1, and Corollary 2.4.1, the following theorem is evident.

Theorem 3.1.6. Let X_c be the controllable subspace of the state space of the system (S). Then

$$X_c^\perp = \sup\{\bar{L}_A^*(V) \cap \left[\left(R(B)\right)^\perp\right]^\nabla\}.$$

This theorem essentially says that the uncontrollable subspace $X_c^\perp$ is the greatest subspace of the state space of the system (S) which is A^T-invariant and is contained in the subspace $\big(R(B)\big)^{\perp}$.

Making use of the relationship between the range and null space of a linear operator E: $V \to V$, given in the following lemma, we can restate Theorem 3.1.6 in terms of $N(B^T)$.

Lemma 3.1.1 Let E: $V \to V$ be a linear operator. Then

$$\big(R(E)\big)^{\perp} = N(E^T).$$

Proof. Let $\underline{y}^T \in N(E^T)$ and $\underline{y} \in R(E)$. Then $\underline{y} = E\underline{x}$ for some $\underline{x} \in V$. But $\langle \underline{y},\underline{y}^T\rangle = \langle E\underline{x},\underline{y}^T\rangle = \langle \underline{x},E^T\underline{y}^T\rangle = 0$ shows that $\underline{y}^T \in \big(R(E)\big)^{\perp}$. Hence $N(E^T) \subset \big(R(B)\big)^{\perp}$.

Now assume $\underline{y}^T \in \big(R(E)\big)^{\perp}$. Then for every $\underline{x} \in V$, $\langle E\underline{x},\underline{y}^T\rangle = 0$. This implies that $\langle \underline{x},A^T\underline{y}^T\rangle = 0$ and hence that $\big(R(E)\big)^{\perp} \subset N(E^T)$.

We can similarly show that the dual to Lemma 3.1.1 also holds, i.e.,

$$R(E^T) = \big(N(E)\big)^{\perp}.$$

Theorem 3.1.7. Let X_c be the controllable subspace of the state space of the system (S). Then

$$X_c^{\perp} = \sup\{\bar{L}_A^{*}(V) \cap \big(N(B^T)\big)^{\nabla}\}.$$

3.2 Output Controllability

For the system (S) the well known criterion for (complete) state controllability in terms of the fixed matrices A and B can be generalized to include the matrices C and D and provide an analogous criterion for (complete) output controllability. We state this result in the following theorem.

Theorem 3.2.1 [48]. The system (S) is output controllable iff the composite $p \times (n+1)m$ matrix

$$[CB \mid CAB \mid CA^2B \mid \cdots \mid CA^{n-1}B \mid D]$$

has rank p.

If we consider a plant without direct transmission, i.e., $D \equiv 0$, then we see that for any matrix B, the output controllable subspace of the output space of the system (S) is the space of the column vectors of the matrix $[CB \mid CAB \mid CA^2B \mid \cdots \mid CA^{n-1}B]$, and in analogy with the idea of complete state controllability, the system (S) is said to be completely output controllable if and only if

$$\operatorname{span}\{CB \mid CAB \mid CA^2B \mid \cdots \mid CA^{n-1}B\} = \mathcal{Y},$$

where $\mathcal{Y}$ is the output space of the system. It is clear that (complete) state controllability can be considered as a special case of (complete) output controllability since with $D = 0$ and $C = I$, the two criteria are identical.

Just as in the case of inputs, we can differentiate between "effective" and "ineffective" outputs in the following sense. If $r(C) < \min(p,n)$, then $\min(p,n) - r(C)$ of the outputs of the system will be linearly dependent on some $r(C)$ of the linearly independent outputs taken from the range space $R(C)$ of the matrix C. Therefore, it is sufficient to consider any two matrices C_1 and C_2 as being output equivalent if and only if $R(C_1) = R(C_2)$. We will consider any matrix C from the equivalence class thus defined as being representative of the whole class.

Let $\sigma_C: L(V) \to L(Y)$ be a lattice morphism induced by the linear mapping $C: V \to Y$ from the n-dimensional linear vector space V into the p-dimensional linear vector space Y. As we have already seen, the mapping σ_C is a join-homomorphism but not in general a meet-homomorphism (Theorem 2.4.1). Consider the join-homomorphic image $\sigma_C\left(L_A^*(V)\right)$ of the lattice of A-invariant subspaces $L_A^*(V)$. We characterize the set $\sigma_C\left(L_A^*(V)\right)$ in the following theorem.

Theorem 3.2.2. The set of all output controllable subspaces of the output space of the system (S), with D = 0, forms a sub-join semilattice $\sigma_C\left(L_A^*(V)\right)$ of the modular lattice $L(Y)$.

Proof. Let $M,N \in L_A^*(V)$ so that $\sigma_C(M)$, $\sigma_C(N) \in \sigma_C\left(L_A^*(V)\right)$. Since σ_C is a join-homomorphism, we have

$$\sigma_C(M) + \sigma_C(N) = \sigma_C(M+N).$$

But $M + N \in L_A^*(V)$ and $\sigma_c(M+N) \in \sigma_c\big(L_A^*(V)\big)$. Hence $\sigma_c(M) + \sigma_c(N) \in \sigma_c\big(L_A^*(V)\big)$. That is, $\sigma_c\big(L_A^*(V)\big)$ contains the join of every pair of its elements and hence is a sub-join semi-lattice of $L(Y)$.

Notice that $\sigma_c\big(L_A^*(V)\big)$ is not in general a sub-meet semi-lattice and hence not a sublattice of $L(Y)$. This is evident from the fact that σ_c is not in general a meet-homomorphism. That is, $M,N \in L(V)$ does not necessarily imply that

$$\sigma_c(M \cap N) = \sigma_c(M) \cap \sigma_c(N)$$

and consequently, $\sigma_c(M)$, $\sigma_c(N) \in \sigma_c\big(L_A^*(V)\big)$ does not necessarily imply that

$$\sigma_c(M) \cap \sigma_c(N) \in \sigma_c\big(L_A^*(V)\big).$$

The unit element of the sub-join semi-lattice $\sigma_c\big(L_A^*(V)\big)$ coincides with the unit element of $L(V)$ if and only if the rank of the matrix C is equal to or greater than the length of the sublattice $[0,Y]$ in $L_A^*(V)$, i.e., $r(C) \geq \dim(Y)$.

Now we will turn to the problem of *output uncontrollability*. The output $\underline{y}(t_0)$ of a dynamical system is said to be uncontrollable at time t_0 if it is not controllable at time t_0. If every output $\underline{y}(t_0)$ of a dynamical system is uncontrollable at time t_0, then the system is said to be *output uncontrollable* at t_0. Furthermore, if a dynamical system is output uncontrollable for all $\underline{y}(t_0)$ and all times t_0, then we say that it is a *(completely) output uncontrollable* system.

It should be pointed out that uncontrollability in the state space does not necessarily imply uncontrollability in the output space with respect to certain inputs. In the system (S) with $D \neq 0$ this is obvious; if $D = 0$, it can be shown that this is still true. In general, (complete) state uncontrollability can be considered as a special case of (complete) output uncontrollability just as (complete) state controllability can be treated as a special case of (complete) output controllability.

The necessary and sufficient condition for output uncontrollability of the system (S), with $D = 0$, is given in the following theorem.

Theorem 3.2.3. The system (S), with $D = 0$, is completely output uncontrollable iff

$$\sigma_c\left(L_A^*(V)\right) = 0 .$$

It is possible to specify the necessary and sufficient condition for output uncontrollability of the system (S) in a particular subspace of the output space of the system as in the following theorem.

Theorem 3.2.4. Let γ: $\sigma_c\left(L_A^*(V)\right) \to L(Y_1)$, with $Y_1 \subset Y$, be a lattice morphism induced by the projection operator P: $Y \to Y_1$. Then the system (S), with $D = 0$, is output uncontrollable in the subspace Y_1 of the output space Y iff

$$(\gamma\circ\sigma_c)\left(L_A^*(V)\right) = 0,$$

where the composite lattice morphism

$$\gamma\circ\sigma_c\colon\ L(V) \to L(Y_1)$$

is induced by the composite mapping

$$P\circ C\colon\ V \to Y_1.$$

3.3 Observability

Earlier we noted that the concept of observability is dual to that of state controllability for the system (S) and using the Duality Theorem of Kalman [40] we can obtain the observability criteria from those of state controllability in a direct manner. Here we will present a brief discussion of the observability criteria which parallels that of state controllability.

For a state $\underline{x}(t) \in R^n$ of the system (S) to be observable, it is necessary and sufficient that $\underline{x}(t)$ be an element of the subspace

$$P_o = \text{span}\{C^T \mid A^TC^T \mid (A^T)^2C^T \mid \cdots \mid (A^T)^{n-1}C^T\}.$$

Therefore, P_o is the observable subspace of the state space of the system (S). Clearly P_o is a subspace of the linear vector space V and thus $P_o \in L(V)$. Furthermore, P_o is an A^T-invariant subspace of V and hence $P_o \in \bar{L}_A^*(V)$. We also observe that P_o is the smallest A^T-invariant subspace of V which contains the range of the matrix C^T, that is,

$$P_o = R(C^T) + A^T R(C^T) + (A^T)^2 R(C^T) + \cdots + (A^T)^{n-1} R(C^T).$$

We summarize the above observations in the following theorem.

Theorem 3.3.1. The subspace X of V is an observable subspace of the state space of the system (S), with D = 0, for some output matrix C iff

$$X = \inf\{\bar{L}_A^*(V) \cap \left(R(C^T)\right)^\Delta\} = \inf\{L_A^*(V) \cap \left[\left(N(E)\right)^\perp\right]^\Delta\}$$

(see Lemma 3.1.1).

Corollary 3.3.1. Let $\{X\}$ be the set of all elements in $\bar{L}_A^*(V)$ which contain $R(C^T) \in L(V)$. Then the observable subspace is $\inf\{X\}$.

This corollary can be equivalently stated as follows.

Corollary 3.3.1'. For some $X \in \bar{L}_A^*(V)$, X is an observable subspace iff $\ell[X,R(C^T)] < \ell[M,R(C^T)]$ for all $M \in \bar{L}_A^*(V)$ and $X \neq M$.

Corollary 3.3.2. For any output matrix $C \in \{C\}_{p\times n}$ and any $X \in \bar{L}_A^*(V)$, X is an observable subspace iff the interval $[X,R(C^T)]$ does not contain any other elements of $\bar{L}_A^*(V)$ except the element X.

The system (S), with D = 0, is said to be *completely observable* if and only if $P_o = V$. As a natural consequence of this definition, we can characterize the set of all output matrices $\{C\}_{p\times n}$ for which the system (S), with D = 0, is completely observable as in the following theorem.

<u>Theorem 3.3.2.</u> For any output matrix $C \in \{C\}_{p\times n}$, the system (S), with D = 0, is completely observable iff the sublattice $[R(C^T), V]$ has only the unit element V of the lattice $L(V)$ in common with the lattice $\bar{L}_A^*(V)$.

<u>Theorem 3.3.3.</u> Let $X \in \bar{L}_A^*(V)$. The set of all output matrices for which X is an observable subspace of the state space of the system (S), with D = 0, consists of the matrices $C \in \{C\}_{p\times n}$ such that

$$R(C^T) \in X^{\nabla} - \bigcup_{\substack{M \subset X \\ M \in \bar{L}_A^*(V)}} X^{\nabla}.$$

<u>Theorem 3.3.4.</u> The single-output dynamical system

$$\dot{\underline{x}} = A\underline{x} + B\underline{u}$$

$$y = \underline{c}^T\underline{x}$$

is observable for some output vector $\underline{c} \in R^n$ iff the lattice $\bar{L}_A^*(V)$ is finite.

By analogy to uncontrollable subspaces, subspaces which are direct complements of an observable subspace X_o are called *unobservable subspaces*. If some state $\underline{x}(t) \in R^n$ is an element of such an unobserable subspace, then $\underline{x}(t)$ cannot be completely reconstructed from a record of the system's inputs and outputs. In particular, the orthogonal complement $X_o^{\perp}$ of the observable subspace. Clearly $X_o^{\perp} \in L_A^*(V)$.

Theorem 3.3.5. Let X be the observable subspace of the state space of the system (S). Then

$$X^{\perp} = \sup\{L_A^*(V) \cap \left(\left(R(C^T)\right)^{\perp}\right)^{\nabla}\},$$

or equivalently,

$$X^{\perp} = \sup\{L_A^*(V) \cap \left(N(C)\right)^{\nabla}\}.$$

This theorem essentially says that the unobservable subspace of the state space of the system (S) is the greatest A-invariant subspace that is contained in the subspace $\left(R(C^T)\right)^{\perp} = N(C)$ (see Lemma 3.1.1).

3.4 State Space Decomposition

In Sections 1 and 3 we noted that the two A-invariant subspaces: the subspace of all controllable states $Q_c = \mathrm{span}\{B \mid AB \mid \cdots \mid A^{n-1}B\}$ and the subspace of all observable states $P_o = \mathrm{span}\{C^T \mid A^TC^T \mid \cdots \mid (A^T)^{n-1}C^T\}$ are intrinsically associated with the state space of the system (S). Kalman [40] has shown that it is always possible to decompose the whole state space into a direct sum of four invariant subspaces.

To accomplish this decomposition, let us uniquely define the A-invariant subspace $\mathcal{W}_1$ by $\mathcal{W}_1 = \mathcal{Q}_c \cap \mathcal{P}_o^{\perp}$ and the other three subspaces by the following relations:

$$\mathcal{Q}_c = \mathcal{W}_1 \oplus \mathcal{W}_2,$$

$$\mathcal{P}_o^{\perp} = \mathcal{W}_1 \oplus \mathcal{W}_3,$$

$$\mathcal{V} = \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4.$$

The subspaces $\mathcal{W}_1$, $\mathcal{W}_2$, $\mathcal{W}_3$, and $\mathcal{W}_4$ can be identified as follows.

$\mathcal{W}_1$ = set of states which are controllable but unobservable.

$\mathcal{W}_2$ = set of states which are controllable and observable.

$\mathcal{W}_3$ = set of states which are uncontrollable and unobservable.

$\mathcal{W}_4$ = set of states which are uncontrollable but observable.

We note that $\mathcal{W}_2$, $\mathcal{W}_3$, and $\mathcal{W}_4$ are not uniquely defined because of the many ways of forming the direct sum. It is clear that the subspaces $\mathcal{W}_1$, $\mathcal{W}_1 \oplus \mathcal{W}_2$, and $\mathcal{W}_1 \oplus \mathcal{W}_3$ are A-invariant.

The preceding discussion can formally be summarized in the following theorem.

<u>Canonical Structure Theorem of Kalman [40]</u>. The state space $\mathcal{V}$ of every dynamical system (S) can be decomposed into four parts as indicated above. Relative to this decomposition of $\mathcal{V}$, the matrices A, B, C, and D have the following canonical forms:

$$\tilde{A} = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} & \tilde{A}_{13} & \tilde{A}_{14} \\ 0 & \tilde{A}_{22} & 0 & \tilde{A}_{24} \\ 0 & 0 & \tilde{A}_{33} & \tilde{A}_{34} \\ 0 & 0 & 0 & \tilde{A}_{44} \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} \tilde{B}_1 \\ \tilde{B}_2 \\ 0 \\ 0 \end{pmatrix},$$

$$\tilde{C} = (0 \quad \tilde{C}_2 \quad 0 \quad \tilde{C}_4), \quad \tilde{D} = D.$$

Now we will briefly discuss some conceptual implications of this theorem.

The theorem essentially says that at every fixed instant t of time, there is a coordinate system in the state space relative to which the components of the state vector can be decomposed into four mutually exclusive parts $\underline{x} = (\underline{x}_1, \underline{x}_2, \underline{x}_3, \underline{x}_4)$, i.e., $\underline{x} = \underline{x}_1 \oplus \underline{x}_2 \oplus \underline{x}_3 \oplus \underline{x}_4$ with $\underline{x}_i \in \mathcal{W}_i$, for i = 1,2,3,4. This decomposition can be achieved in many ways, but the number of state variables in each part is the same for any such decomposition.

The zeros in the matrices $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ can be heuristically justified as follows:

1. Since $\underline{x}_3(t)$ and $\underline{x}_4(t)$ are uncontrollable states, it should not be possible to control them directly by the input $\underline{u}$ or indirectly by allowing terms involving controllable states to enter into the equations for $\underline{x}_3(t)$ and $\underline{x}_4(t)$. Hence, zeros appear in the last two "B" blocks and also in the positions $\tilde{A}_{31}$, $\tilde{A}_{32}$, $\tilde{A}_{41}$, and $\tilde{A}_{42}$ of the $\tilde{A}$ matrix.

2. $\underline{x}_1(t)$ and $\underline{x}_3(t)$ are unobservable states, it should not be possible to identify them from the knowledge of the system output; i.e., their contribution to the system output must be zero. This explains the zeros in the $\tilde{C}$ matrix. Furthermore, their contribution to the states $\underline{x}_2(t)$ and $\underline{x}_4(t)$ should be zero since those states will appear in the output expression. Hence we have zeros in the positions $\tilde{A}_{21}$, $\tilde{A}_{23}$, $\tilde{A}_{41}$, and $\tilde{A}_{43}$ of the $\tilde{A}$ matrix.

As it was pointed out in Chapter I, the problem of state space decomposition for a linear dynamical system has also been investigated by other authors [24,29,69,70].

Next we present a lattice-theoretic representation of Kalman's Canonical Structure Theorem.

Let $X_c \in L_A^*(V)$ and $X_o \in \bar{L}_A^*(V)$ denote the controllable and observable subspaces of the state space of the system (S), respectively. Then

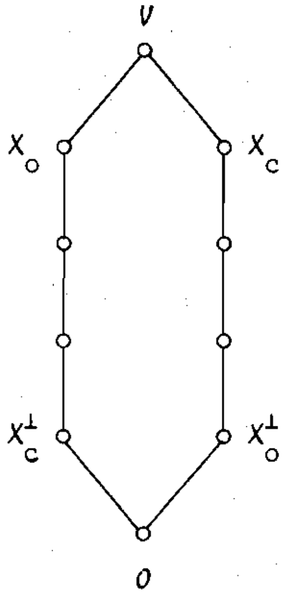
$$V = (X_c \cap X_o) + (X_c^{\perp} \cap X_o) + (X_c \cap X_o^{\perp}) + (X_c^{\perp} \cap X_o^{\perp}),$$

where $X_c^{\perp} \in \bar{L}_A^*(V)$ and $X_o^{\perp} \in L_A^*(V)$ are the uncontrollable and unobservable subspaces of the state space of the system (S).

A two-chain lattice diagram of this decomposition for a completely controllable and completely observable linear dynamical system is shown below:

$X_o, X_c \in \bar{L}_A^*(V)$

$X_o \cap X_c \in \bar{L}_A^*(V)$



$X_c, X_o^\perp \in L_A^*(V)$

$X_c \cap X_o^\perp \in L_A^*(V)$

Notice that if $L_A^*(V)$ is a chain, then $X_o \subseteq X_c^\perp$ or $X_c^\perp \subseteq X_o$ and $X_c \subseteq X_o^\perp$ or $X_o^\perp \subseteq X_c$. Furthermore, $X_c^\perp \cap X_o^\perp$ and $X_c \cap X_o$ are proper A- or A^T-invariant subspaces, i.e., if $X_c^\perp \cap X_o^\perp$ or $X_c \cap X_o$ is an element of $L_A^*(V)$ or $\bar{L}_A^*(V)$, then it is either 0 or V.

CHAPTER IV

GENERALIZED CONTROLLABILITY SUBSPACES

4.1 Generalized State Controllability Subspaces

In Chapter III we noticed that the concept of invariance of a linear subspace under a linear operator is intimately linked with the important structural properties of controllability and observability of a linear dynamical system. In this chapter, we will consider a generalization of the concept of invariance by introducing two additional lattices with special structures: a lattice $K(B)$ of invariant subspaces of $L(V)$ which contains the lattice of A-invariant subspaces $L_A^*(V)$ as a sublattice; and a lattice $\bar{K}(B)$ which also contains $L_A^*(V)$ as a sublattice and is dually related to $K(B)$. This generalization consequently leads to an extension of the controllability and observability subspaces for the linear dynamical system (S). A somewhat similar extension was introduced, in a vector space setting, by Basile and Marro [6] and also independently by Wonham and Morse [68] and was successfully used for the identification and characterization of new types of controllability and observability subspaces such as perfect output controllability subspace, functional output controllability subspace, unknown-input state observability subspace, and functional input observability subspace. These subspaces and their associated properties were briefly discussed in Chapter I. In the latter part of this chapter, we will consider some applications of the concept of generalized invariance and its underlying implications to certain feedback compensation problems.

Let

$$K(\mathrm{B}) = \{X \in L(V) \colon \mathrm{A}X \subseteq X + R(\mathrm{B}) \text{ for fixed linear operators A and B}\}$$

From this definition, the following results are obvious:

i. $K(\mathrm{B})$ is a lattice with zero element 0 and unit element V.

ii. If $R(\mathrm{B}) = 0$, or $R(\mathrm{B}) \subseteq X$, then $K(\mathrm{B}) = L_{\mathrm{A}}^{*}(V)$.

iii. If $R(\mathrm{B}) = V$, then $K(\mathrm{B}) = L(V)$.

iv. If $X + R(\mathrm{B}) = V$, then $K(\mathrm{B}) = L(V)$.

v. If $X + R(\mathrm{B}) \in L_{\mathrm{A}}^{*}(V)$, then $X \in K(\mathrm{B})$.

Theorem 4.1.1. $K(\mathrm{B})$ is closed under the join operation with respect to $L(V)$.

Proof. Let $X_1, X_2 \in K(\mathrm{B})$. Then we have

$$\mathrm{A}X_1 \subseteq X_1 + R(\mathrm{B}),$$

$$\mathrm{A}X_2 \subseteq X_2 + R(\mathrm{B}),$$

so that

$$\mathrm{A}X_1 + \mathrm{A}X_2 = \mathrm{A}(X_1 + X_2) \subseteq X_1 + X_2 + R(\mathrm{B}).$$

Note that $K(\mathrm{B})$ is not closed under the meet operation with respect to

$L(V)$, i.e., $X_1, X_2 \in K(B)$ does not necessarily imply that $A(X_1 \cap X_2) \subseteq (X_1 \cap X_2) + R(B)$ since $A(X_1 \cap X_2)$ is not, in general, a subspace of $X_1 \cap X_2$. However, if we assume X_1 and X_2 to be elements of the lattice $L_A^*(V)$, then $K(B)$ will be closed under both the meet and join operations with respect to $L(V)$. We will formalize this observation in the following theorem whose simple proof is omitted.

Theorem 4.1.2. $K(B)$ contains $L_A^*(V)$ as a sublattice.

The following theorem will prove useful in the study of feedback compensation problems.

Theorem 4.1.3 (cf. [68]). Let $X \in L(V)$. Then there exists a matrix F such that $X \in L_{A+BF}^*(V)$ iff $X \in K(B)$.

In order to prove this theorem, first we need a lemma.

Lemma 4.1.1 [68]. Let $\underline{h}_i \in R^n$, $\underline{g}_i \in R^m$, $i = 1,2,\ldots,k$, and let $H = (\underline{h}_1, \underline{h}_2, \ldots, \underline{h}_k)$, $G = (\underline{g}_1, \underline{g}_2, \ldots, \underline{g}_k)$. Then there exists an $m \times n$ matrix F such that $F\underline{h}_i = \underline{g}_i$, $i = 1,2,\ldots,k$, iff $N(H) \subseteq N(G)$. F always exists if the $\underline{h}_i$ are linearly independent.

Proof of Theorem 4.1.3. Suppose $X \in K(B)$ and let $\{\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_k\}$ be a basis of X. Then $A\underline{x}_i = B\underline{u}_i + \underline{w}_i$ for some $\underline{u}_i \in R^m$ and $\underline{w}_i \in X$. Choose the $m \times n$ matrix F, by Lemma 4.1.1 such that

$$F\underline{x}_i = -\underline{u}_i, \quad i = 1,2,\ldots,k;$$

then

$$\left.\begin{aligned} A\underline{x}_i - B\underline{u}_i &= \underline{w}_i \\ A\underline{x}_i + B(-\underline{u}_i) &= \underline{w}_i \\ A\underline{x}_i + BF\underline{x}_i &= \underline{w}_i \\ (A+BF)\underline{x}_i &= \underline{w}_i \end{aligned}\right\} i = 1,2,\ldots,k.$$

Hence

$$(A+BF)\mathcal{X} \subseteq \mathcal{X}.$$

To prove the necessary part of the theorem, suppose that there exists an $m \times n$ matrix F such that

$$(A+BF)\mathcal{X} \subseteq \mathcal{X}.$$

This implies that

$$A\underline{x}_i + BF\underline{x}_i = \underline{w}_i,\ \underline{w}_i \in \mathcal{X}, \quad i = 1,2,\ldots,k.$$

$$A\underline{x}_i = -BF\underline{x}_i + \underline{w}_i.$$

$$A\underline{x}_i = B(-F\underline{x}_i) + \underline{w}_i.$$

Therefore,

$$A\mathcal{X} \subseteq \mathcal{R}(B) + \mathcal{X},$$

i.e.,

$$X \in K(B).$$

<u>Theorem 4.1.4</u>. Suppose the matrices B and F are fixed and let $M \in L(V)$. Then the set

$$W = \{X\colon X \in L^{*}_{A+BF}(V) \text{ and } X \subseteq M\}$$

is contained in the set

$$M^{\nabla} \cap K(B).$$

Proof. Let $X \in W$. Then $X \subseteq M$ and $X \in M^{\nabla}$. By Theorem 4.1.3 $X \in L^{*}_{A+BF}(V)$ if and only if $X \in K(B)$. Hence

$$X \in M^{\nabla} \cap K(B).$$

<u>Theorem 4.1.5</u>. The set $M^{\nabla} \cap K(B)$, if nonempty, has a maximal element.

Proof. Let $X_1 \in M^{\nabla} \cap K(B)$. If X_1 is the only element in the set, then X_1 is maximal. If $X_1, X_2 \in M^{\nabla} \cap K(B)$, then $X_1 + X_2 \in M^{\nabla} \cap K(B)$, since $X_1, X_2 \in M^{\nabla}$ and $K(B)$ is closed under the join operation with respect to $L(V)$. By induction, $M^{\nabla} \cap K(B)$ has a maximal element $\sum_{i=1}^{k} X_i$, $X_i \in M^{\nabla} \cap K(B)$, $i = 1,2,\ldots,k$, for some finite k.

Next we introduce another lattice $\bar{K}(B)$ which is dually related to $K(B)$.

Let

$$\bar{K}(B) = \{X \in L(V)\colon A(X \cap R(B)) \subseteq X \text{ for fixed linear operators A and B}\}.$$

From this definition, the following results are obvious:

i. $\bar{K}(B)$ is a lattice with zero element 0 and unit element V.

ii. If $R(B) = V$ or $R(B) \subseteq X$, then $\bar{K}(B) = L_A^*(V)$.

Theorem 4.1.6. $\bar{K}(B)$ is closed under the meet operation with respect to $L(V)$.

Proof. Let $X_1, X_2 \in \bar{K}(B)$. Then we have

$$A(X_1 \cap R(B)) \subseteq X_1,$$

$$A(X_2 \cap R(B)) \subseteq X_2,$$

so that

$$[A(X_1 \cap R(B))] \cap [A(X_2 \cap R(B))] = A((X_1 \cap X_2) \cap R(B)) \subseteq X_1 \cap X_2.$$

It is clear that, in general, $X_1, X_2 \in \bar{K}(B)$ does not necessarily imply the relationship $A((X_1+X_2) \cap R(B)) \subseteq X_1 + X_2$, indicating that $\bar{K}(B)$ is not closed under the join operation with respect to $L(V)$. However, if we restrict X_1 and X_2 to be elements of the lattice $L_A^*(V)$, then $\bar{K}(B)$, like

$K(B)$, becomes closed under both the meet and join operations with respect to $L(V)$. We formalize this statement in the following theorem whose simple proof is omitted.

Theorem 4.1.7. $\bar{K}(B)$ contains $L_A^*(V)$ as a sublattice.

It follows directly from Theorem 4.1.1 and Theorem 4.1.6 that the set $K(B) \cup \bar{K}(B)$ is a sublattice of $L(V)$ in which the meet and join operations are the same as in $L(V)$.

Now we will concentrate on further detailed characterization of the lattice $\bar{K}(B)$.

a. If $X \in L_A^*(V)$, then $X \in \bar{K}(B)$ for all $R(B)$.

b. If $R(B)$ and X are 1-dimensional subspaces, then for all $X \neq R(B)$, $A(X \cap R(B)) = A\ 0 \subseteq X$ and the condition $A(X \cap R(B)) \subseteq X$ does not hold if and only if $X = R(B)$ and $R(B) \notin L_A^*(V)$.

c. If $R(B)$ is a 1-dimensional subspace and X is a 2-dimensional subspace, then if

1. $R(B) \subseteq X$ and $X \notin L_A^*(V)$ then $X \notin \bar{K}(B)$. In other words, in such a case if $X \in (R(B))^{\Delta}$, then $X \notin \bar{K}(B)$.

2. If $R(B) \subseteq X$ and $R(B) \in L_A^*(V)$, then $X \in \bar{K}(B)$.

3. If $R(B) \nsubseteq X$ and $X \cap R(B) = 0$, then $X \in \bar{K}(B)$. As an important consequence of this case we observe that, in general, if $R(B) \nsubseteq X$, then $X \in \bar{K}(B)$. In other words, if X is a direct relative complement of $R(B)$ relative to some space, then $X \in \bar{K}(B)$.

We can summarize the above observations for a 1-dimensional subspace $R(B) \in L(V)$ and arbitrary subspace $X \in L(V)$ as follows:

i. If $X \in L_A^*(V)$, then $X \in \bar{K}(B)$ for all $R(B)$.

ii. If $R(B) \subseteq X$ and $R(B) \in L_A^*(V)$, then $X \in \bar{K}(B)$.

iii. If $R(B) \nsubseteq X$ and $X \cap R(B) = 0$, then $X \in \bar{K}(B)$.

iv. If $R(B) \subseteq X$ and $X \notin L_A^*(V)$, then $X \notin \bar{K}(B)$.

Theorem 4.1.8. Let $M \in L(V)$. The set $M^\nabla \cap \bar{K}(B)$, if nonempty, has a minimal element.

Proof. Since $0 \in M^\nabla$ and $0 \in \bar{K}(B)$, 0 is the minimal element of the set $M^\nabla \cap K(B)$. The rest of the proof is similar to that of Theorem 4.1.5.

In order to establish the relationship between the lattices $K(B)$ and $\bar{K}(B)$, we need the following lemma:

Lemma 4.1.2. For any $X,Y \in L(V)$, $AX \subseteq Y$ iff $A^TY^\perp \subseteq X^\perp$.

Proof. Suppose $AX \subseteq Y$ and let $\underline{y} \in A^TY^\perp = \{A^T\underline{z}: \underline{z}\in Y^\perp\}$. Then we have $\underline{y} = A^T\underline{z}$ for $\underline{z} \in Y^\perp$. Since $AX = \{A\underline{x}: \underline{x}\in X\} \subseteq Y$, we have $\langle\underline{z},A\underline{x}\rangle = 0$. But $\langle\underline{z},A\underline{x}\rangle = \langle A^T\underline{z},\underline{x}\rangle = 0$. This implies that $A^T\underline{z} \in X^\perp$, hence $\underline{y} = A^T\underline{z} \in X^\perp$. Thus $AX \subseteq Y \Rightarrow A^TY^\perp \subseteq X^\perp$.

The converse part of the theorem can be proved in a similar manner.

Theorem 4.1.9. If $AX \subseteq X + R(B)$, then $A^T\left(X^\perp \cap (R(B))^\perp\right) \subseteq X^\perp$.

Proof. By Lemma 4.1.2, $AX \subseteq X + R(B) \Rightarrow A^T(X+R(B))^T \subseteq X^\perp$. By Lemma 2.4.1,

$$\left(\mathcal{X}+\mathcal{R}(\mathrm{B})\right)^{\perp} = \mathcal{X}^{\perp} \cap \left(\mathcal{R}(\mathrm{B})\right)^{\perp}.$$

Hence

$$\mathrm{A}\mathcal{X} \subseteq \mathcal{X} + \mathcal{R}(\mathrm{B}) \Rightarrow \mathrm{A}^{\mathrm{T}}\left[\mathcal{X}^{\perp}\cap\left(\mathcal{R}(\mathrm{B})\right)^{\perp}\right] \subseteq \mathcal{X}^{\perp}.$$

Theorem 4.1.10. If $\mathrm{A}\left(\mathcal{X}\cap\mathcal{R}(\mathrm{B})\right) \subseteq \mathcal{X}$, then $\mathrm{A}^{\mathrm{T}}\mathcal{X}^{\mathrm{T}} \subseteq \mathcal{X}^{\perp} + \left(\mathcal{R}(\mathrm{B})\right)^{\perp}$.

Proof. By Lemma 4.1.2, $\mathrm{A}\left(\mathcal{X}\ \mathcal{R}(\mathrm{B})\right) \subseteq \mathcal{X} \Rightarrow \mathrm{A}^{\mathrm{T}}\mathcal{X}^{\perp} \subseteq \left(\mathcal{X}\cap\mathcal{R}(\mathrm{B})\right)^{\perp}$. By Lemma 2.4.1,

$$\left(\mathcal{X}\ \mathcal{R}(\mathrm{B})\right)^{\perp} = \mathcal{X}^{\perp} + \left(\mathcal{R}(\mathrm{B})\right)^{\perp}.$$

Hence

$$\mathrm{A}\left(\mathcal{X}\cap\mathcal{R}(\mathrm{B})\right) \subseteq \mathcal{X} \Rightarrow \mathrm{A}^{\mathrm{T}}\mathcal{X}^{\perp} \subseteq \mathcal{X}^{\perp} + \left(\mathcal{R}(\mathrm{B})\right)^{\perp}.$$

Theorem 4.1.11. The lattices $\mathcal{K}(\mathrm{B})$ and $\bar{\mathcal{K}}(\mathrm{B})$ are dually isomorphic.

Proof. This result follows directly from the one-to-one correspondence established between $\mathcal{K}(\mathrm{B})$ and $\bar{\mathcal{K}}(\mathrm{B})$ in Theorem 4.1.9 and Theorem 4.1.10.

The next theorem provides a method for computing the least element in $\bar{\mathcal{K}}(\mathrm{B})$ containing a given element of $\mathcal{L}(\mathcal{V})$.

Theorem 4.1.12 (cf. [6]). Suppose $\mathcal{X} \in \mathcal{L}(\mathcal{V})$ is given and let

$$X_m = \inf\{\tilde{X}: \tilde{X}\epsilon\tilde{K}(B)\} \in X^{\Delta}.$$

Then $X_m = Y_{n-1}$, where Y_{n-1} is defined by the following recursive relationship:

$$Y_0 = X,$$

$$Y_i = X + A\big(Y_{i-1}\cap R(B)\big), \quad i = 1,2,\ldots,n-1.$$

Proof. By successive evaluation we can easily show that the above sequence of elements of the lattice $L(V)$ is equivalent to the following sequence:

$$Y_0 = X,$$

$$Y_i = Y_{i-1} + A\big(Y_{i-1}\cap R(B)\big), \quad i = 1,2,\ldots,n-1.$$

From this sequence it follows immediately that

$$Y_{i-1} \subseteq Y_i,$$

$$X \subseteq Y_i, \quad i = 1,2,\ldots,n-1.$$

If for a certain value of the index k, $Y_k = Y_{k-1}$, then $Y_i = Y_{k-1}$ for all $i \geq k-1$. To see this consider

$$\mathcal{Y}_k = \mathcal{Y}_{k-1} + A\left(\mathcal{Y}_{k-1} \cap \mathcal{R}(B)\right).$$

Since

$$\mathcal{Y}_k = \mathcal{Y}_{k-1}, \quad \mathcal{Y}_k = \mathcal{Y}_k + A\left(\mathcal{Y}_k \cap \mathcal{R}(B)\right).$$

But

$$\mathcal{Y}_{k+1} = \mathcal{Y}_k + A\left(\mathcal{Y}_k \cap \mathcal{R}(B)\right).$$

Hence

$$\mathcal{Y}_{k-1} = \mathcal{Y}_k = \mathcal{Y}_{k+1}.$$

Continuing in this manner, we see that $\mathcal{Y}_i = \mathcal{Y}_{k-1}$ for all $i \geq k-1$. It also follows that

$$A\left(\mathcal{Y}_{k-1} \cap \mathcal{R}(B)\right) \subseteq \mathcal{Y}_{k-1}$$

since

$$A\left(\mathcal{Y}_{k-1} \cap \mathcal{R}(B)\right) \subseteq \mathcal{Y}_{k-1} + A\left(\mathcal{Y}_{k-1} \cap \mathcal{R}(B)\right) = \mathcal{Y}_k = \mathcal{Y}_{k-1}.$$

The above result indicates the fact that if the dimension of $\mathcal{X}$ is at least one, then in order to find a subspace $\mathcal{Y}_j$ such that $\mathcal{Y}_j \in \bar{\mathcal{R}}(B)$ for some specified operator A and $\mathcal{R}(B) \in \mathcal{L}(\mathcal{V})$, it is sufficient to stop the sequence at the term $\mathcal{Y}_{n-1}$.

To complete the proof, we must show that

$$\mathcal{X}_m = \inf\{\tilde{\mathcal{X}}\colon \tilde{\mathcal{X}} \in \bar{\mathcal{K}}(B)\}.$$

That is, we need to show that there does not exist $X_1 \in X^{\Delta}$ such that

$$\tilde{X}_1 \in \bar{K}(B)$$

and

$$\tilde{X}_1 \in X_m^{\nabla}.$$

To accomplish this, we will use a contradiction argument. Suppose there exists $\tilde{X}_1$ such that $\tilde{X}_1 \supseteq X = Y_0$, $A(\tilde{X}_1 \cap R(B)) \subseteq \tilde{X}_1$, and $\tilde{X}_1 \subset X_m = Y_{n-1}$. Then

$$\begin{aligned} \tilde{X}_1 &\supseteq Y_0 + A(Y_0 \cap R(B)) = Y_1, \\ &\vdots \\ \tilde{X}_1 &\supseteq Y_{n-2} + A(Y_{n-2} + R(B)) = Y_{n-1}. \end{aligned}$$

This is obviously a contradiction.

<u>Theorem 4.1.13 (cf. [6]).</u> Suppose $X \in L(V)$ is given and let

$$X_m = \sup\{\tilde{X}\colon \tilde{X} \in K(B)\} \in X^{\nabla}.$$

Then

$$X_m = Y_{n-1}^{\perp},$$

where

$$Y_0 = X^{\perp},$$

$$Y_i = X^{\perp} + A^{T}\left\{Y_{i-1} \cap \left(R(B)\right)^{\perp}\right\}, \quad i = 1,2,\ldots,n-1.$$

Proof. This follows from Lemma 4.1.2, Theorem 4.1.9, Theorem 4.1.10 and Theorem 4.1.12.

The above theorem provides a method for constructing a basis for the greatest element of $K(B)$ contained in a specified element of $L(V)$.

As an example of how some of these concepts may be efficiently applied in the study of control systems, next we will reformulate and prove a theorem due to Basile and Marro [6].

Theorem 4.1.14 (cf. [6]). Given $X \in L(V)$, a trajectory of the system (S) starting from a point $\underline{x}_o \in X$ can be controlled on X in any finite interval of time iff

$$\underline{x}_o \in X_m = \sup\{\tilde{X}: \tilde{X} \in K(B)\} \in X^{\nabla}.$$

Proof. Suppose $\underline{x}_o \in X_m$. Since $X_m \in K(B)$, by Theorem 4.1.3 there exists a matrix F such

$$X_m \in L^{*}_{A+BF}(V).$$

This implies that there exists a feedback compensation $\underline{u} = F\underline{x} + G\underline{v}$ such that the trajectory can be controlled on X.

Conversely, if the trajectory can always be controlled on the subspace X, then the system is controllable on X and the set of all controllable states of the system forms a subspace X_1 of X. By the definition of a controllability subspace, it follows that $X_1 \subseteq X_m$.

4.2 Feedback Compensation

In Chapter III we showed that if for some subspace X of the state space of the system (S) the condition $AX \subseteq X$ holds, i.e., if the subspace X is A-invariant, then we can always choose an input matrix B such that the subspace X is the (maximal) controllable subspace of the state space of the system. For example, it suffices to choose the input matrix B such that $R(B) = X$.

Let us now assume that X is a subspace of the state space of the system (S) which is *not* A-invariant; otherwise, X is arbitrary. The question of interest then is: Can we find an appropriate state feedback compensation $\underline{u}(t) = F\underline{x}(t) + G\underline{v}(t)$ such that the subspace X becomes a controllable subspace of the state space of the system (S') for some input matrix BG? Now the system (S') is given by the equations

$$\begin{aligned} \dot{\underline{x}}(t) &= (A+BF)\underline{x}(t) + BG\underline{v}(t) \\ \underline{y}(t) &= (C+DF)\underline{x}(t) + DG\underline{v}(t). \end{aligned} \qquad (S')$$

It follows immediately from the preceding discussion that X is a controllable subspace of the state space of the system (S') for some input matrix BG if and only if there exists a feedback compensation matrix F such that

$$(A+BF)X \subseteq X, \qquad (4.1)$$

i.e., if and only if there exists a matrix F such that $X \in L^*_{A+BF}(V)$

and $x = R(BG)$. More specifically, if the input matrix BG is fixed, then X is a controllable subspace of the state space of the system (S') if and only if the condition (4.1) holds and X is the smallest (A+BF)-invariant subspace of V which contains $R(BG)$, i.e.,

$$X = \inf\{L^{*}_{A+BF}(V) \cap \left(R(BG)\right)^{\Delta}\}.$$

This condition can be more neatly reformulated by noting that $R(BG) \subseteq R(B)$ and the following theorem.

<u>Theorem 4.2.1 (cf. [68]).</u> If $X = \inf\{L^{*}_{A}(V) \cap \left(R(\hat{B})\right)^{\Delta}\}$ for some $R(\hat{B}) \in \left(R(B)\right)^{\nabla}$, then

$$X = \inf\{L^{*}_{A}(V) \cap \left(R(B) \cap X\right)^{\Delta}\}.$$

Conversely, if

$$X = \inf\{L^{*}_{A}(V) \cap \left(R(B) \cap X\right)^{\Delta}\},$$

then there exists a matrix G such that

$$X = \inf\{L^{*}_{A}(V) \cap \left(R(BG)\right)^{\Delta}\}.$$

Proof. $X = \inf\{L^{*}_{A}(V) \cap \left(R(\hat{B})\right)^{\Delta}\}$ implies $R(\hat{B}) \subseteq X$ so that

$$R(\hat{B}) \subseteq R(B) \cap X,$$

and thus

$$X = \inf\{L_A^*(V) \cap (R(\hat{B}))^\Delta\} \subseteq \inf\{L_A^*(V) \cap (R(B) \cap X)^\Delta\}.$$

To show the reverse inclusion, note that since $X \in L_A^*(V)$, i.e., $AX \subseteq X$,

$$A(R(B) \cap X) \subseteq X;$$

and by induction

$$A^j(R(B) \cap X) \subseteq X, \quad j = 1,2,\ldots\ .$$

Therefore,

$$\inf\{L_A^*(V) \cap (R(B) \cap X)^\Delta\} \subseteq X.$$

Hence

$$X = \inf\{L_A^*(V) \cap (R(B) \cap X)^\Delta\}.$$

To prove the converse of the theorem, let $\underline{b}_i$, $i = 1,2,\ldots,m$, be the *ith* column of the matrix B and let $\{\underline{x}_j\}$, $j = 1,2,\ldots,m'$, be a basis of $R(B) \cap X$. Then

$$\underline{x}_j = \sum_{i=1}^{m} g_{ij}\underline{b}_i, \quad j = 1,2,\ldots,m',$$

for suitable g_{ij}, where $G = [g_{ij}]$ is an $m \times m'$ matrix. This obviously shows the existence of G such that

$$X = \inf\{L_A^*(V) \cap \left(R(BG)\right)^\Delta\}.$$

By the above theorem, if for fixed matrices A and B and a given subspace X there exists a matrix F such that

$$X = \inf\{L_{A+BF}^*(V) \cap \left(R(B) \cap X\right)^\Delta\},$$

then X is called a *controllability subspace* of the pair (A,B). Observe that $X = 0$ and $X = \inf\{L_A^*(V) \cap \left(R(B)\right)^\Delta\}$ are controllability subspaces.

In order to characterize controllability subspaces, we need the following lemmas.

<u>Lemma 4.2.1</u>. For all $X \in L(V)$,

$$AX + R(B) = (A+BF)X + R(B),$$

where A, B, and F are fixed matrices.

Proof. Since $AX + R(B) = AX + B(FX) + R(B)$ and $B(FX) \subseteq R(B)$, we have $BFX + R(B) = R(B)$ and hence

$$AX + R(B) = AX + BFX + R(B) = AX + R(B).$$

<u>Lemma 4.2.2</u>. Let $X \in L_{A+BF}^*(V)$ and $\tilde{X} \in X^\nabla$. Then

$$X \cap R(B) + (A+BF)\tilde{X} = X \cap \left(A\tilde{\ } + R(B)\right),$$

where A, B, and F are fixed matrices.

Proof. By Lemma 4.2.1,

$$A\tilde{X} + R(B) = (A+BF)\tilde{X} + R(B),$$

which implies that

$$X \cap \left(AX+R(B)\right) = X \cap [(A+BF)\tilde{X}+R(B)].$$

Since $L(V)$ is a modular lattice, we have

$$\begin{aligned} X \cap \left(A\tilde{X}+R(B)\right) &= X \cap [(A+BF)\tilde{X}+R(B)] \\ &= (A+BF)\tilde{X} + X \cap R(B). \end{aligned}$$

Lemma 4.2.3 (cf. [68]). If $X \in L^{*}_{A+BF}(V)$, then

$$\sum_{j=1}^{i} (A+BF)^{j-1}\left(R(B)\cap X\right) = X_i,$$

$$i = 1,2,\ldots,n,$$

where

$$X_i = \begin{cases} 0 & \text{if } i = 0, \\ X \cap \left(AX_{i-1}+R(B)\right) & \text{if } i = 1,2,\ldots,n. \end{cases} \tag{4.3}$$

Proof. To prove this lemma we will use an induction argument. Equation (4.2) is true for i = 1 since

$$\sum_{j=1}^{1} (A+BF)^{0}\left(\mathcal{R}(B)\cap\mathcal{X}\right) = \mathcal{X}_1$$

$$\left(\mathcal{R}(B)\cap\mathcal{X}\right) = \mathcal{X}_1.$$

But from (4.3),

$$\begin{aligned}\mathcal{X}_1 &= \mathcal{X} \cap \left(A\mathcal{X}_o+\mathcal{R}(B)\right)\\ &= \mathcal{X} \cap \left(0+\mathcal{R}(B)\right)\\ &= \mathcal{X} \cap \mathcal{R}(B).\end{aligned}$$

Now we assume that (4.2) is true for k - 1 and show that it holds for k. Consider

$$\sum_{j=1}^{k-1} (A+BF)^{j-1}\left(\mathcal{R}(B)\cap\mathcal{X}\right) = \mathcal{X}_{k-1}$$

$$[(A+BF)^{0}+(A+BF)^{1}+(A+BF)^{2}+\ldots+(A+BF)^{k-2}]\left(\mathcal{R}(B)\cap\mathcal{X}\right) = \mathcal{X}_{k-1}.$$

Multiplying both sides of this equation by (A+BF) we get

$$[(A+BF)+(A+BF)^{2}+(A+BF)^{3}+\ldots+(A+BF)^{k-1}]\left(\mathcal{R}(B)\cap\mathcal{X}\right) = (A+BF)\mathcal{X}_{k-1}.$$

Thus

$$\sum_{j=2}^{k} (A+BF)^{j-1}\left(\mathcal{R}(B)\cap\mathcal{X}\right) = (A+BF)\mathcal{X}_{k-1}.$$

Adding $(A+BF)^0\left(\mathcal{R}(B)\cap\mathcal{R}(B)\right)$ to both sides of this equation yields

$$\sum_{j=1}^{k} (A+BF)^{j-1}\left(\mathcal{R}(B)\cap\mathcal{X}\right) = \left(\mathcal{R}(B)\cap\mathcal{X}\right) + (A+BF)\mathcal{X}_{k-1}.$$

By Lemma 4.2.2,

$$\left(\mathcal{R}(B)\cap\mathcal{X}\right) + (A+BF)\mathcal{X}_{k-1} = \mathcal{X} \cap \left(A\mathcal{X}_{k-1}+\mathcal{R}(B)\right).$$

From (4.3),

$$\mathcal{X} \cap \left(A\mathcal{X}^{(k-1)}+\mathcal{R}(B)\right) = \mathcal{X}_k.$$

Therefore,

$$\sum_{j=1}^{k} (A+BF)^{j-1}\left(\mathcal{R}(B)\cap\mathcal{X}\right) = \mathcal{X}_k.$$

This completes the proof of the lemma.

<u>Theorem 4.2.2 (cf. [68])</u>. Let $\mathcal{X} \in \mathcal{L}(\mathcal{V})$. Then $\mathcal{X}$ is a controllability subspace of the pair (A,B) if and only if $\mathcal{X} \in \mathcal{K}(B)$ and $\mathcal{X} = \tilde{\mathcal{X}}$, where $\tilde{\mathcal{X}}$ is the minimal subspace such that $\tilde{\mathcal{X}} = \mathcal{X} \cap \left(A\tilde{\mathcal{X}}+\mathcal{R}(B)\right)$. Furthermore,

$$\tilde{X} = X_\delta,$$

where $\delta = \dim(X)$ and

$$X_0 = 0,$$

$$X_i = X \cap \left(AX_{i-1} + R(B)\right), \quad i = 1,2,\ldots,n.$$

Proof. Suppose $X \in L(V)$ is a controllability subspace, i.e.,

$$X = \inf\{L^*_{A+BF}(V) \cap \left(R(B)\cap X\right)^\Delta\}.$$

This implies that

$$X \in L^*_{A+BF}(V).$$

By Theorem 4.1.3,

$$AX \subseteq X + R(B) \quad \text{if and only if} \quad X \in L^*_{A+BF}(V).$$

By Lemma 4.2.3,

$$X = \sum_{j=1}^{n} (A+BF)^{j-1}\left(R(B)\cap X\right) = X_n = X_\delta.$$

Conversely, if $X = X_n$, then

$$X = \sum_{j=1}^{n} (A+BF)^{j-1}\left(R(B)\cap X\right)$$

for every F such that $(A+BF)X \subseteq X$.

It remains to show that $X = X \cap \left(A\tilde{X}+R(B)\right)$ has the minimal solution X_δ. By induction on i in

$$X_0 = 0$$

$$X_i = X \cap \left(AX_{i-1}+R(B)\right), \quad i = 1,2,\ldots,n,$$

it is seen that $X_i \subseteq \tilde{X}$, $i = 1,2,\ldots$, for every solution $\tilde{X}$ of $\tilde{X} = X \cap \left(A\tilde{X}+R(B)\right)$, and that the sequence X_i is monotone nondecreasing. Hence, there exists a $\mu \leq \delta$ such that $X_i = X_\mu$ for $i \geq \mu$, in particular, $X_\delta \subseteq X$ and X_δ satisfies $X = X \cap \left(A\tilde{X}+R(B)\right)$.

4.3 Generalized Output Controllability Subspaces

In this section we will present a complete characterization of a controllable output subspace of the output space of the system (S). This development is related to and based on the concept of a controllability subspace discussed in the previous section, and generalizes the notions of output controllability and invariance treated in Section 3.2. This idea was originally introduced by Morse [53] in a vector space framework. Here we will consider a lattice-theoretic presentation of his theory. As a matter of fact, we will imitate his format of presentation.

Let X, U, and Y denote the state, input, and output spaces associated with the system (S). Linear mappings and their matrices will be denoted by the same symbol. The notation $\{A|R(B)\}$ is defined as

$$\{A|\mathcal{R}(B)\} = \inf\{\mathcal{L}_A^*(\mathcal{V}) \cap (\mathcal{R}(B))^\Delta\}.$$

Let $\mathcal{Z} = \mathcal{X} \oplus \mathcal{Y}$.

Below we define some lattice morphisms, the linear mappings that induce these morphisms, and indicate their particular assignments.

$$\pi_1: \mathcal{L}(\mathcal{Z}) \to \mathcal{L}(\mathcal{X}), \quad P: \mathcal{Z} \to \mathcal{X}, \quad \underline{z} \mapsto P\underline{z},$$

$$\pi_2: \mathcal{L}(\mathcal{Z}) \to \mathcal{L}(\mathcal{Y}), \quad Q: \mathcal{Z} \to \mathcal{Y}, \quad \underline{z} \mapsto Q\underline{z},$$

P denotes the projection of $\mathcal{Z}$ onto $\mathcal{X}$ along $\mathcal{Y}$. Q denotes the projection of $\mathcal{Z}$ onto $\mathcal{Y}$ along $\mathcal{X}$.

$$\alpha: \mathcal{L}(\mathcal{X}) \to \mathcal{L}(\mathcal{Z}), \quad A: \mathcal{X} \to \mathcal{Z}, \quad \underline{x} \mapsto A\underline{x},$$

$$\bar{\alpha} = \alpha \circ \pi_1: \mathcal{L}(\mathcal{Z}) \to \mathcal{L}(\mathcal{X}), \quad \bar{A}: \mathcal{Z} \to \mathcal{Z}, \quad \underline{z} \mapsto AP\underline{z},$$

$$\bar{\beta}: \mathcal{L}(\mathcal{U}) \to \mathcal{L}(\mathcal{Z}), \quad \bar{B}: \mathcal{U} \to \mathcal{Z}, \quad \underline{u} \mapsto B\underline{u},$$

$$\gamma: \mathcal{L}(\mathcal{X}) \to \mathcal{L}(\mathcal{Z}), \quad C: \mathcal{X} \to \mathcal{Z}, \quad \underline{x} \mapsto C\underline{x},$$

$$\bar{\gamma} = \gamma \circ \pi_1: \mathcal{L}(\mathcal{Z}) \to \mathcal{L}(\mathcal{Z}), \quad \bar{C}: \mathcal{Z} \to \mathcal{Z}, \quad \underline{z} \mapsto CP\underline{z},$$

$$\bar{\delta}: \mathcal{L}(\mathcal{U}) \to \mathcal{L}(\mathcal{Z}), \quad \bar{D}: \mathcal{U} \to \mathcal{Z}, \quad \underline{u} \mapsto D\underline{u}.$$

The following relationships are immediate consequences of the definition of the projection operators P and Q:

$$\bar{A} = \bar{A}P = P\bar{A},$$

$$\bar{C} = Q\bar{C} = \bar{C}P,$$

$$\bar{B} = P\bar{B},$$

$$\bar{D} = Q\bar{D}.$$

Below A, B, C, D and their extensions $\bar{A}$, $\bar{B}$, $\bar{C}$, $\bar{D}$ will be denoted by the same symbols A, B, C, D. The interpretation, in each instance, will be clear from the context.

Now consider the feedback control

$$\underline{u}(t) = F\underline{x}(t) + G\underline{w}(t), \tag{4.4}$$

where $\underline{w} \in \mathcal{W} \in \mathcal{L}(\mathcal{V})$. In connection with this control law, we define the following lattice morphisms and linear mappings inducing them:

$$\sigma\colon\ \mathcal{L}(\mathcal{W}) \to \mathcal{L}(\mathcal{U}),\quad G\colon \mathcal{W} \to \mathcal{U},\quad \underline{w} \mapsto G\underline{w},$$

$$\phi\colon\ \mathcal{L}(\mathcal{X}) \to \mathcal{L}(\mathcal{Z}),\quad F\colon \mathcal{X} \to \mathcal{Z},\quad \underline{x} \mapsto F\underline{x},$$

$$\bar{\phi} = \phi\circ\pi_1\colon\ \mathcal{L}(\mathcal{Z}) \to \mathcal{L}(\mathcal{U}),\quad \bar{F}\colon \mathcal{Z} \to \mathcal{U},\quad \underline{z} \mapsto FP\underline{z}.$$

The extended mapping $\bar{F}$ is associated with the matrix F in (4.4), i.e.,

$$\bar{F} = FP. \tag{4.5}$$

The system (S) under the influence of the feedback control law (4.4) becomes

$$\begin{aligned} \dot{\underline{x}}(t) &= (A+BF)\underline{x}(t) + BG\underline{w}(t) \\ \underline{y}(t) &= (C+DF)\underline{x}(t) + DG\underline{w}(t). \end{aligned} \tag{S*}$$

The largest subspace $X_c \in X^\nabla$, $X \in L(V)$, which $\underline{w}$ can control is given by

$$X_c = \inf\{L^*_{A+BF}(V) \cap (R(BG))^\Delta\}. \tag{4.6}$$

If for fixed A, B, and $X_c \subseteq X \in L(V)$, a pair (F,G) exists for which (4.6) is true, then X_c is called a controllability subspace. The existence condition for X_c is given in Theorem 4.2.2. In the system (S*) we observe that the application of the feedback control (4.4) to the system (S) affects not only state controllability, but output controllability as well. Thus if Y_c is the largest subspace of Y which $\underline{w}$ can completely control, then clearly

$$Y_c = (C+DF)X_c + R(DG), \tag{4.7}$$

with X_c given by (4.6). Hence, the characterization of a controllable output subspace reduces to the following problem: For fixed Y_c, Y, find conditions for the existence of a pair (F,G) and a subspace $X_c \subseteq X \in L(V)$ for which both (4.6) and (4.7) are satisfied.

Conditions (4.6) and (4.7) can be simplified.

Lemma 4.3.1 (cf. [53]). Let the lattice morphisms

$$\tilde{\alpha} = \alpha \circ \pi_1 : L(Z) \to L(Z), \quad \tilde{A}: Z \to Z,$$

$$\tilde{\beta}: \quad L(U) \to L(Z), \quad \tilde{B}: U \to Z$$

be fixed and suppose $\tilde{A} = \tilde{A}P$. If for fixed G,

$$X_c \equiv \{P\tilde{A}|PR(\tilde{B}G)\}, \quad Y_c \equiv Q\tilde{A}X_c + QR(\tilde{B}G), \tag{4.8}$$

then

$$\begin{aligned} X_c &= P\{\tilde{A}|R(\tilde{B}) \cap (X_c+Y_c)\}, \\ Y_c &= Q\{\tilde{A}|R(\tilde{B}) \cap (X_c+Y_c)\}. \end{aligned} \tag{4.9}$$

Conversely, if X_c and Y_c are fixed subspaces satisfying (4.9), there exists a G for which (4.8) is true.

Proof. From the relation $\tilde{A} = \tilde{A}P$ follow the identities

$$P\{\tilde{A}|R(\tilde{B}G)\} \equiv \{P\tilde{A}|PR(\tilde{B}G)\},$$

$$Q\{\tilde{A}|R(\tilde{B}G)\} \equiv Q\tilde{A}\{P\tilde{A}|PR(\tilde{B}G)\} + QR(\tilde{B}G). \tag{4.10}$$

Let $\tilde{M} \equiv \{\tilde{A}|R(\tilde{B}G)\}$ and $\tilde{N} \equiv \{\tilde{A}|R(\tilde{B}) \cap (X_c+Y_c)\}$.

If (4.8) is true, (4.10) provides $X_c = P\tilde{M}$ and $Y_c = Q\tilde{M}$. Thus $\tilde{M} \subseteq X_c + Y_c$. But $R(\tilde{B}G) \subseteq R(\tilde{B}) \cap \tilde{M} \subseteq R(\tilde{B}) \cap (X_c+Y_c)$, so $\tilde{M} \subseteq \{\tilde{A}|R(\tilde{B}) \cap (X_c+Y_c)\} = \tilde{N}$. Now $R(\tilde{B}) \cap (X_c+Y_c) \subseteq X_c + Y_c$ and if $W \subseteq X_c + X_c$, $\tilde{A}W \subseteq X_c + Y_c$. Thus $\tilde{N} \subseteq X_c + Y_c$. Hence $X_c = P\tilde{M} \subseteq P\tilde{N} \subseteq X_c$ and $Y_c \subseteq Q\tilde{M} \subseteq Q\tilde{N} \subseteq Y_c$. Therefore,

$$X_c = P\tilde{N} \quad \text{and} \quad Y_c = Q\tilde{N}.$$

Conversely, if (4.9) holds, $X_c = P\tilde{N}$ and $Y_c = Q\tilde{N}$. By Theorem 4.2.1, there exists a G such that $\tilde{N} = \{\tilde{A}|R(\tilde{B}G)\}$. It follows from this and (4.10) that (4.8) is true.

Recalling constraint (4.5) and applying Lemma 4.3.1 with

$$\tilde{A} \equiv (A+BF) + (C+DF),$$

$$\tilde{B} \equiv B + D,$$

we can restate the existence problem as follows:

Let A, B, C, D, and $Y_c \subseteq Y$ be fixed. Find the conditions for the existence of a lattice morphism

$$\phi: \mathcal{L}(\mathcal{U}) \to \mathcal{L}(\mathcal{Z}), \quad F: \mathcal{U} \to \mathcal{Z}$$

and a subspace $\tilde{N} \subseteq \mathcal{Z}$ such that

$$\bar{F} = FP, \tag{4.11}$$

$$\mathcal{Y}_c = Q\tilde{N}, \tag{4.12}$$

$$\tilde{N} = \{A+C+(B+D)F \mid R(B+D) \cap (P\tilde{N}+\mathcal{Y}_c)\}. \tag{4.13}$$

If F and $\tilde{N}$ satisfying (4.11)-(4.13) exist, $\mathcal{Y}_c$ will be called a controllable output subspace of (A,B,C,D). The subspace $\mathcal{X}_c \equiv P\tilde{N}$ will be called a *generator* of $\mathcal{Y}_c$.

Let $\bar{\tilde{N}}(\tilde{M})$ be the maximal controllability subspace of the pair (A+B,C+D) contained in a fixed subspace $\tilde{M} \subseteq \mathcal{Z}$, i.e.,

$$\bar{\tilde{N}}(\tilde{M}) = \sup \inf\{\mathcal{L}^*_{A+C+(B+D)F}(\mathcal{V}) \cap \left(R(B+D)\cap\bar{\tilde{N}}(\tilde{M})\right)^{\Delta}\} \in \tilde{M}^{\Delta},$$

$M \subseteq \mathcal{Z} \in \mathcal{L}(\mathcal{V})$. Theorem 4.2.2 provides a procedure for determining $\bar{\tilde{N}}(\tilde{M})$. The solution to the above existence problem is now presented.

<u>Theorem 4.3.1 (cf. [53])</u>. Let (A,B,C,D), $\mathcal{X}_c \subseteq \mathcal{X}$, and $\mathcal{Y}_c \subseteq \mathcal{Y}$ be fixed. Then $\mathcal{Y}_c$ is a controllable output subspace with generator $\mathcal{X}_c$ if and only if

$$\mathcal{Y}_c = Q\bar{\tilde{N}}(\mathcal{X}_c+\mathcal{Y}_c), \quad \mathcal{X}_c = P\bar{\tilde{N}}(\mathcal{X}_c+\mathcal{Y}_c). \tag{4.14}$$

Proof. If $\mathcal{Y}_c$ is a controllable output subspace with generator $\mathcal{X}_c$, then there exist F and $\tilde{\mathcal{N}}$ satisfying (4.11)-(4.13) with $\mathcal{X}_c = P\tilde{\mathcal{N}}$. Thus,

$$\begin{aligned}\left(A+C+(B+D)F\right)(\mathcal{X}_c+\mathcal{Y}_c) &= \left(A+C+(B+D)F\right)P(\mathcal{X}_c+\mathcal{Y}_c)\\ &= \left(A+C+(B+D)F\right)\tilde{\mathcal{N}}\\ &\subseteq \tilde{\mathcal{N}} \subseteq \mathcal{X}_c + \mathcal{Y}_c.\end{aligned}$$

That is, $(\mathcal{X}_c+\mathcal{Y}_c) \in \mathcal{L}^*_{A+C+(B+D)F}(\mathcal{V})$. But the controllability subspace $\tilde{\mathcal{N}}$ is maximal relative to $\mathcal{X}_c + \mathcal{Y}_c$, that is, $\tilde{\mathcal{N}} = \bar{\bar{\mathcal{N}}}(\mathcal{X}_c+\mathcal{Y}_c)$ and (4.14) is true.

Conversely, if (4.14) holds, $\mathcal{Y}_c$ will be a controllable output subspace with $\mathcal{X}_c$ as a generator if there exists a matrix F satisfying (4.11) for which

$$\bar{\bar{\mathcal{N}}}(\mathcal{X}_c+\mathcal{Y}_c) = \{A+C+(B+D)F|\mathcal{R}(B+D) \cap (\mathcal{X}_c+\mathcal{Y}_c)\}. \qquad (4.15)$$

Since $\bar{\bar{\mathcal{N}}} \equiv \tilde{\mathcal{N}}(\mathcal{X}_c+\mathcal{Y}_c)$ is a controllability subspace, by Theorem 4.2.2.,

$$(A+C)\bar{\bar{\mathcal{N}}} \subseteq \mathcal{R}(B+D) + \bar{\bar{\mathcal{N}}},$$

i.e.,

$$\bar{\bar{\mathcal{N}}} \in \mathcal{K}(B+D).$$

This and (4.14) provide

$$(A+C)\mathcal{X}_c = (A+C)\bar{\tilde{\mathcal{N}}} \subseteq \mathcal{R}(B+D) + \bar{\tilde{\mathcal{N}}} \subseteq \mathcal{R}(B+D) + (\mathcal{X}_c+\mathcal{Y}_c).$$

By Theorem 4.1.3, there exists a matrix $\tilde{F}$ such that

$$\left(A+C+(B+D)\tilde{F}\right)\mathcal{X}_c \subseteq \mathcal{X}_c + \mathcal{Y}_c.$$

If $F \equiv \tilde{F}P$, then

$$\left(A+C+(B+D)F\right)(\mathcal{X}_c+\mathcal{Y}_c) = \left(A+C+(B+D)\tilde{F}\right)\mathcal{X}_c \subseteq \mathcal{X}_c + \mathcal{Y}_c,$$

so that

$$(\mathcal{X}_c+\mathcal{Y}_c) \in \mathcal{L}^*_{A+C+(B+D)F}(\mathcal{V}).$$

It is clear that (4.15) holds for this choice of F. In addition, F satisfies (4.11) as required.

In general, a fixed output controllable subspace may have many generators. It is useful, therefore, to characterize controllable output subspaces with a hypothesis involving generators. This is easily accomplished.

Corollary 4.3.1 (cf. [53]). Let (A,B,C,D), $\mathcal{Y}_c \subseteq \mathcal{Y}$ be fixed. Then $\mathcal{Y}_c$ is a controllable output subspace if and only if there exists a controllability subspace $\tilde{\mathcal{N}}$ of (A+C,B+D) such that

$$\mathcal{Y}_c = Q\tilde{N}. \tag{4.16}$$

If (4.16) holds, $X_c \equiv P\tilde{N}$ is a generator of $\mathcal{Y}_c$.

Proof. Necessity is obvious. Take $\tilde{N} = \tilde{\bar{N}}(X_c+\mathcal{Y}_c)$ and apply Theorem 4.3.1. For sufficiency note that $\tilde{N} \subseteq X_c + \mathcal{Y}_c$. Thus $\tilde{N} \subseteq \tilde{\bar{N}}(X_c+\mathcal{Y}_c)$, so

$$\mathcal{Y}_c = Q\tilde{N} \subseteq Q\tilde{\bar{N}}(X_c+\mathcal{Y}_c) \subseteq \mathcal{Y}_c$$

and

$$X_c = P\tilde{N} \subseteq P\tilde{\bar{N}}(X_c+\mathcal{Y}_c) \subseteq X_c.$$

It follows that X_c and $\mathcal{Y}_c$ satisfy Theorem 4.3.1 which, in turn, provides the desired result.

CHAPTER V

SOME STRUCTURE THEOREMS AND NONINTERACTING CONTROLS

In this chapter we will present the lattice-theoretic analogues and/or extensions of a number of existing results in linear dynamical systems theory. This collection represents a small subset of the theoretical and practical aspects of linear systems to which our lattice-theoretic approach can be applied. Here the manner of presentation will differ considerably from previous chapters in that here we will provide only the essential definitions and terminology needed for the understanding of the concepts embodied in the assertions of the theorems. No attempt has been made to construct lattice-theoretic proofs for the theorems or explore their implicational and applicability aspects in this chapter since this, although undoubtedly feasible, would exceed the scope of this thesis.

The primary objective of including this chapter is threefold:

1. To reiterate the applicability and unifying potential of our lattice-theoretic approach to a wide range of analysis and synthesis problems in linear dynamical systems theory and applications.

2. To report some interesting and useful recent results in the literature of linear control systems discipline.

3. To provide challenge, motivation, and ample directions for further research in lattice-theoretic concepts in the area of linear control systems.

It is indeed interesting to compare in each case our formulation with the original statement of the concept appearing in the reference indicated next to our theorem and evaluate their intuitive appeal and generality.

5.1 Some New Structural Properties of Linear Time-Invariant Dynamical Systems

Definition 5.1.1. The locus of the points of the output subspace of the output space of the system (S), with D = 0, which can be kept constant for a finite time interval by means of a suitable control function is called the *constant output subspace* and is denoted by $\mathcal{Y}_{con}$.

The following theorem provides an expression for this subspace.

Theorem 5.1.1 (cf. [6]).

$$\mathcal{Y}_{con} = C\mathcal{X}, \quad \mathcal{X} = A^{\dagger}\left[\left(\mathcal{X}_m+\mathcal{R}(B)\right) \cap \mathcal{R}(A)\right] + \mathcal{N}(A),$$

where $A^{\dagger}$ denotes the pseudoinverse of A, and

$$\mathcal{X}_m = \sup\{\tilde{\mathcal{X}}\colon \tilde{\mathcal{X}}\in\mathcal{K}(B)\} \in \left(\mathcal{N}(C)\right)^{\nabla}.$$

If the matrix A is nonsingular, the above expression for $\mathcal{Y}_{con}$ can be written as

$$\mathcal{Y}_{con} = C\mathcal{X}, \quad \mathcal{X} = A^{-1}\left(\mathcal{X}_m+\mathcal{R}(B)\right).$$

Definition 5.1.2. A subspace $Y^{(i)} \in L(Y)$ is said to be a *perfect output controllability subspace with respect to the ith derivative* if there exists at least a subspace $X^{(i)} \in L(V)$ such that $Y^{(i)} = CX^{(i)}$ and that, for every initial state $\underline{x}_0 \in X^{(i)}$, it is possible, by means of proper bounded and measurable control functions, to follow in $Y^{(i)}$ any trajectory arbitrarily given in the class of functions which admit i*th* derivative with respect to time, while the state evolves on $X^{(i)}$.

The following properties are direct consequences of the above definition:

1. Every subspace $X^{(i)} \in L(V)$, corresponding to a subspace of perfect output controllability $Y^{(i)}$ is an element of the lattice $K(B)$.

2. A subspace of perfect output controllability with respect to the i*th* derivative is a subspace of perfect output controllability with respect to the derivatives of any order greater than i.

3. The sum of two subspaces of perfect output controllability with respect to the i*th* derivative is a subspace of perfect output controllability with respect to the i*th* derivative.

Theorem 5.1.2 (cf. [9]). An element $X^{(n)}$ of the lattice $K(B)$ corresponds in the state space of the system (S), with D = 0, to a subspace $Y^{(n)}$ of perfect output controllability with respect to the n*th* derivative iff the following condition holds.

$$X^{(n)} \subseteq X^{(n)} \cap \hat{X} + N(C),$$

where

$$\hat{X} = \inf\{\tilde{X}\cap N(C)\in K(B)\} \in \left(R(B)\right)^{\Delta}.$$

The next theorem provides an iterative procedure for computing the greatest subspace $X^{(i)} \in K(B)$ contained in a given subspace $W \in L(V)$ corresponding to a subspace $Y^{(i)}$ of perfect output controllability in the sense of Definition 5.1.2.

Theorem 5.1.3 (cf. [9]). Given $W \in L(V)$, the element

$$X^{(i)} = \sup\{\tilde{X}^{(i)}:\ \tilde{X}^{(i)}\in K(B)\} \in W^{\nabla}$$

satisfying the condition $X^{(i)}\subseteq X^{(i)}\cap Z_{i-1}+N(C)$, where Z_{i-1} is given by the recursive relationship:

$$Z_0 = R(B),$$

$$Z_k = R(B) + A(Z_{k-1}\cap X^{(i)}\cap N(C),\ k = 1,2,\ldots,i-1,$$

is the last term $(X^{(i)}=S_{n-1})$ of the sequence of elements of the lattice $L(V)$:

$$S_0 = W,$$

$$S_k = \sup\{\tilde{S}:\ \tilde{S}\in K(B)\} \in \left(S_{k-1} \cap Z_{k,i-1} + N(C)\right)^{\nabla},$$

where $Z_{k,i-1}$ is given by the recursive relationship:

$$Z_{k,o} = R(B),$$

$$Z_{k,h} = R(B) + A\left(Z_{k,h-1} \cap S_{k-1} \cap N(C)\right), \quad h=1,2,\ldots,i-1.$$

<u>Definition 5.1.3.</u> A dynamical system is *completely unknown-input state observable by means of differentiators* if it is possible to determine its state $\underline{x}(t)$ when an arbitrarily short output segment $\underline{y}(\tau)$, $\tau \in [t,t+\epsilon]$ is given.

Note that because of linearity, when unknown-input state observability by means of differentiators is not complete, it is generally possible to observe the orthogonal projection of the state on a subspace, while the complementary orthogonal projection is not observable at all. Such a subspace will be called *subspace of unknown-input state observability by means of differentiators*.

<u>Definition 5.1.4.</u> A dynamical system is *invertible* or *completely functionally input observable* if it is possible to determine its input $\underline{u}(t)$ when an arbitrarily short output segment $\underline{y}(\tau)$, $\tau \in [t,t+\epsilon]$ and the state $\underline{x}(t)$ are given.

Again, when functional input observability is not complete, only the orthogonal projection of the input on a subspace, which will be called *subspace of functional input observability*, can be determined.

<u>Theorem 5.1.4 (cf. [10]).</u> The unknown-input state observability subspace of the state space of the system (S), with D = 0, is given by

$$\mathcal{X}_m = \inf\{\tilde{\mathcal{X}}:\ A^T\big(\tilde{\mathcal{X}}\cap N(B^T)\big) \subseteq \tilde{\mathcal{X}}\} \supseteq R(C^T). \tag{5.1}$$

By the duality relationship proved in Theorem 4.1.9 and Theorem 4.1.10, it follows from the above theorem that the state can be recognized within a vector on

$$\mathcal{X}_m = \sup\{\tilde{\mathcal{X}}:\ \tilde{\mathcal{X}}\epsilon K(B)\} \in \big(N(C)\big)^{\nabla}$$

since $\big(R(B)\big)^{\perp} = N(B^T)$ and $\big(R(C^T)\big)^{\perp} = N(C)$ (see Lemma 3.1.1).

Theorem 5.1.5 (cf. [10]). The functional input observability subspace of the state space of the system (S), with D = 0, is given by

$$\mathcal{X}_f = B^T\mathcal{X}_m,$$

where $\mathcal{X}_m$ is given by (5.1).

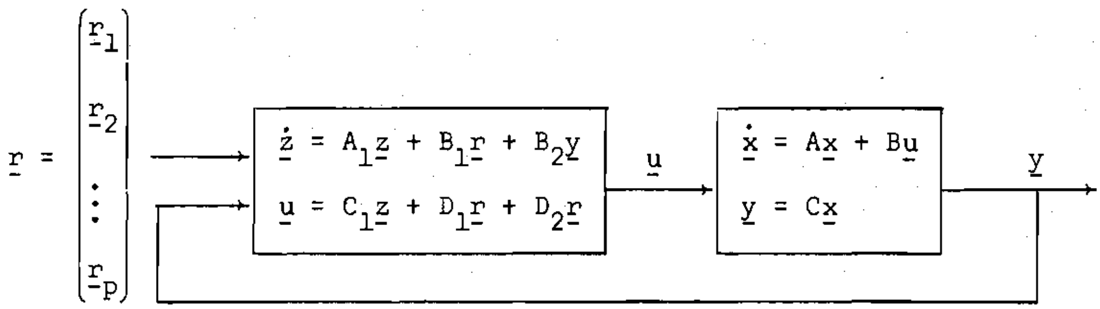
Corollary 5.1.1 (cf. [10]). For the system (S), with D = 0, the subspace of unknown-input state unobservability by means of differentiators is given by

$$\mathcal{X}_m^{\perp} = \sup\{\tilde{\mathcal{X}}:\ \tilde{\mathcal{X}}\epsilon K(B)\} \in \big(N(C)\big)^{\nabla} \tag{5.2}$$

$$= \sup\{\tilde{\mathcal{X}}:\ \tilde{\mathcal{X}}\epsilon K(B)\} \in \Big(\big(R(C^T)\big)^{\perp}\Big)^{\nabla}.$$

5.2 Noninteraction and Decoupling Problems

Consider the system (S), with D = 0, in the following noninteracting control arrangement



Controller Controlled System

where the controller state $\underline{z} \in R^k$, where k and the matrices of appropriate orders A_1, B_1, B_2, C_1, D_1, D_2 are completely arbitrary.

Definition 5.2.1. Given a set of subspaces $\mathcal{Y}_1,\ldots,\mathcal{Y}_p$ of the output space $\mathcal{Y}$, which intersect only at the origin, the controller shown in the above figure is said to be *noninteracting* on them if for every $i = 1,2,\ldots,p$, starting from a rest condition and varying in any way only the input $\underline{r}_i$, an output trajectory which completely belongs to the subspace $\mathcal{Y}_i$, is obtained.

Let $\tilde{\underline{x}} = \begin{pmatrix} \underline{x} \\ \underline{z} \end{pmatrix} \in R^{n+k}$ be an augmented state vector, which accounts also for the state variables of the controller. Equation of the controlled system together with the equations of the controller can be written in the compact form

$$\dot{\tilde{\underline{x}}} = \tilde{A}\tilde{\underline{x}} + \tilde{B}\underline{r}$$
$$\tilde{\underline{y}} = \tilde{C}\tilde{\underline{x}}, \tag{$\tilde{S}$}$$

where the matrices $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ can be easily computed in terms of the matrices A, B, C, A, B_1, B_2, C_1, D_1, D_2.

<u>Theorem 5.2.1 (cf. [8]).</u> The system ($\tilde{S}$) is noninteracting on the subspaces $\mathcal{Y}_1,\ldots,\mathcal{Y}_p$ iff the following condition holds:

$$\tilde{C}\mathcal{X}_m \subseteq \mathcal{Y}_i, \quad i = 1,2,\ldots,p,$$

with

$$\mathcal{X}_m = \inf\{\tilde{\mathcal{X}}: \tilde{\mathcal{X}} \in \mathcal{L}_A^*(\mathcal{V})\} \in \left(\mathcal{R}(\tilde{B}_i)\right)^{\nabla},$$

where $\tilde{B}_i$, i = 1,2,...,p, is the partition of the columns of the matrix $\tilde{B}$ corresponding to the partition $\underline{r}_i$, i = 1,2,...,p, of the input vector $\underline{r}$.

<u>Theorem 5.2.2 (cf. [8]).</u> For every output space $\mathcal{Y}_i$, i = 1,2, ...,p, the greatest subspace $\mathcal{Y}_i' \subseteq \mathcal{Y}_i$ which can be reached by means of trajectories completely belonging to $\mathcal{Y}_i$ is defined by the following relationships:

$$\mathcal{X}_{0i} = \sup\{\tilde{\mathcal{X}}: \tilde{\mathcal{X}} \in \mathcal{K}(B)\} \in (C^{-1*}\mathcal{Y}_i)^{\nabla},$$
$$\mathcal{X}_i' = \inf\{\tilde{\mathcal{X}}: \tilde{\mathcal{X}} \in \mathcal{L}_{A+BF}^*(\mathcal{V})\} \in \left(\mathcal{X}_{0i} \cap \mathcal{R}(B)\right)^{\Delta},$$

$$\mathcal{Y}_i' = C\mathcal{X}_i',$$

where $C^{-1*}\mathcal{Y}_i = \{\underline{y}: C\underline{y}\epsilon\mathcal{Y}_i\}$, and F is any matrix such that $(A+BF)\mathcal{X}_{0i} \subseteq \mathcal{X}_{0i}$.

Corollary 5.2.1 (cf. [8]). For the noninteraction and complete controllability on the subspaces $\mathcal{Y}_i$, $i = 1,2,\ldots,p$, the condition $\mathcal{Y}_i' = \mathcal{Y}_i$ is necessary.

Consider the output equation

$$\underline{y}(t) = C\underline{x}(t), \tag{5.3}$$

with

$$C = \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_k \end{bmatrix},$$

where C_i is of dimension $p_i \times n$, $i = 1,2,\ldots,k$, $k \geq 2$, and $p_1 + p_2 + \ldots + p_k = p$. Then (5.3) can be written

$$\underline{y}_i = C_i\underline{x}(t), \quad i = 1,2,\ldots,k,$$

where $\underline{y}_i$ is a p_i-vector. That is, the system output $\underline{y}$ consists of k subvectors

$$\underline{y} = \begin{bmatrix} \underline{y}_1 \\ \underline{y}_2 \\ \vdots \\ \underline{y}_k \end{bmatrix}.$$

Algebraically, the structure of $\underline{y}$ is equivalent to the assumption

$$\mathcal{Y} = \mathcal{Y}_1 \oplus \mathcal{Y}_2 \oplus \dots \oplus \mathcal{Y}_k,$$

where $\mathcal{Y}_i$ is the output subspace associated with $\underline{y}_i$, $i = 1,2,\dots,k$.

The vector $\underline{y}_i$ may be regarded as physically significant groups of scalar output variables. It may therefore be desirable to control completely each of the output vectors $\underline{y}_i$ individually, without affecting the behavior of the remaining $\underline{y}_j$, $j \neq i$. This end is to be achieved by linear state-variable feedback together with the assignment of a suitable group of control inputs to each $\underline{y}_i$. That is, in the system (S), with D = 0, we set

$$\underline{u} = F\underline{x} + \sum_{i=1}^{k} G_i \underline{v}_i .$$

For $\underline{v}_i$ to control $\underline{y}_i$ completely, we must have

$$\mathcal{Y}_i = C_i \mathcal{X}_i = \mathcal{R}(C_i), \quad i = 1,2,\dots,k, \tag{5.4}$$

where

$$X_i = \inf\{L^*_{A+BF}(V) \cap \left(X_i \cap R(BG_i)\right)^\Delta\}, \tag{5.5}$$

i.e., X_i is the controllability subspace of the pair $(A+BF, BG_i)$ associated with $\underline{v}_i$, $i = 1,2,\ldots,k$. Since the *ith* control $\underline{v}_i$ is to leave the output $\underline{y}_j$, $j \neq i$, unaffected, we require also

$$C_j X_i = 0, \quad j \neq i. \tag{5.6}$$

By Theorem 4.2.1 and the fact that $R(BF) \subseteq R(B)$ the expressions

$$X = \inf\{L^*_{A+BF}(V) \cap \left(R(BF)\right)^\Delta\}$$

and

$$X = \inf\{L^*_{A+BF}(V) \cap \left(R(B) \cap X\right)^\Delta\}$$

are equivalent. Using this result, we can state the decoupling problem as follows: Given A, B and $N(C_1),\ldots,N(C_k)$, find a matrix F and controllability subspaces $X_1, X_2, \ldots, X_k$, with the properties

$$X_i = \inf\{L^*_{A+BF}(V) \cap \left(R(B) \cap X_i\right)^\Delta\}, \quad i = 1,\ldots,k, \tag{5.7}$$

$$X_i + N(C_i) = V, \quad i = 1,2,\ldots,k, \tag{5.8}$$

$$X_i \in \Big(\bigcap_{j \neq i} N(C_j)\Big)^\nabla, \; i = 1,2,\ldots,k. \tag{5.9}$$

Here (5.8) and (5.9) are equivalent, respectively, to (5.4) and (5.6).

Let

$$\tilde{X}_i = \sup\inf\{L^*_{A+BF}(V) \cap [R(B)\cap\tilde{X}_i]^\Delta\} \in \big(\bigcap_{j\neq i} N(C_j)\big)^\nabla.$$

We now consider the determination of necessary and sufficient conditions for the existence of a solution to (5.7)-(5.9) in two special cases.

1. $r(C) = n$. This assumption is equivalent to

$$\bigcap_{i=1}^{k} N(C_i) = 0.$$

That is, there is a one-to-one mapping of state variables into output variables.

Theorem 5.2.3 (cf. [68]). If $\bigcap_{i=1}^{k} N(C_i) = 0$, then the problem (5.7)-(5.9) has a solution iff

$$\tilde{X}_i + N(C_i) = V, \quad i = 1,\ldots,k.$$

2. $r(B) = k$. This assumption is equivalent to

$$\dim[R(B)] = k.$$

Theorem 5.2.4 (cf. [68]). If $\dim[R(B)] = k$, then the problem (5.7)-(5.9) has a solution iff

$$\tilde{X}_i + N(C_i) = V, \quad i = 1,\ldots,k,$$

and

$$R(B) = \sum_{i=1}^{k} R(B) \cap \tilde{X}_i.$$

Furthermore, if F, $R(X_1),\ldots,R(X_k)$ is any solution, then

$$X_i = \tilde{X}_i, \quad i = 1,2,\ldots,k.$$

Recall that the above discussion of the state feedback decoupling of output variables was exclusively for the case when D = 0 in the system (S). However, if D ≠ 0, then using the results and notation of Section 4.3 and the notation introduced above, the decoupling problem for the system (S) can be treated as follows. In this case the decoupling requirement, i.e., choosing the matrices F and G_i so that $\underline{v}_i$ can control $\underline{y}_i$ without influencing $\underline{y}_j$ for $j \neq i$, dictates that

$$X_i = \inf\{L^*_{A+BF}(V) \cap \left(R(BG_i)^\Delta\right\}; \ R(BG_i) \in L(V), \quad i = 1,2,\ldots,k,$$

$$Y_i = (C+DF)X_i + R(DG_i); \ R(BG_i) \in L(V), \ i = 1,2,\ldots,k.$$

The decoupling problem can be stated as follows.

Find conditions for the existence of a lattice morphism

$$\bar{\phi} = \phi \circ \pi_1 : L(Z) \to L(U)$$

and subspaces $\tilde{N}_i$, $i = 1,2,\ldots,k$, such that

$$\tilde{N}_i = \inf\{L^*_{A+C+(B+D)F}(V) \cap (R(B+D)\cap\tilde{N}_i)^\Delta\}, \quad i = 1,2,\ldots,k,$$

$$Y_i = Q\tilde{N}_i, \quad k = 1,2,\ldots,k,$$

with the notation as in Section 4.3.

The solution to the decoupling problem is given in the following theorem.

Theorem 5.2.5 (cf. [53]). If $\dim[R(B+D)] = k$, then the state feedback decoupling problem has a solution iff

$$Q\bar{\bar{N}}_i(Y_i+X) = Y_i, \quad i = 1,2,\ldots,k,$$

and

$$R(B+D) = \sum_{i=1}^{k} \left(R(B+D) \cap \bar{\bar{N}}_i(Y_i+X)\right).$$

Furthermore, if F, $\tilde{N}_1,\ldots,\tilde{N}_k$ is any solution, then

$$\tilde{N}_i = \bar{\bar{N}}_i(Y_i+X), \quad i = 1,2,\ldots,k,$$

where

$$\bar{\bar{N}}_i(Y_i+X) = \sup \inf\{L^*_{A+C+(B+D)F}(V) \cap (R(B+D)\cap\bar{\bar{N}}_i(Y_i+X))^\Delta\} \in (Y_i+X)^\Delta,$$

$$i = 1,2,\ldots,k.$$

CHAPTER VI

CONCLUSIONS AND RECOMMENDATIONS

In this thesis, for the purpose of investigating some structural properties of linear time-invariant multivariable dynamical systems from an entirely different perspective, a finite-dimensional linear vector space V is "replaced" by a complete conplemented atomic modular lattice, namely the lattice $L(V)$ of linear subspaces of V, and subsequently it is shown that this point of view is conducive to the development of a new theory for a certain broad class of linear dynamical systems. The structures and properties of the sublattice $L^*(V)$ of $L(V)$, the lattice of invariant subspaces of V, are shown to be intimately related with many important characteristics of the system

$$\begin{aligned} \dot{\underline{x}}(t) &= A\underline{x}(t) + B\underline{u}(t) \\ \underline{y}(t) &= C\underline{x}(t) + D\underline{u}(t). \end{aligned} \qquad (S)$$

Various structures of the lattices of invariant subspaces of V with respect to different types of linear operator are investigated and described. Necessary and sufficient conditions for the finiteness of these lattices are given and existence of linear operators which generate desired structures of invariant subspaces is discussed. Properties of the lattice of invariant subspaces are then related to state controllability, output controllability, and observability properties of linear time-invariant dynamical systems. In particular, the set of input matrices for which the system (S) is completely state controllable and the set of output matrices for which the system (S) is completely observable are characterized in terms of the sublattices of $L(V)$.

A lattice-theoretic representation of Kalman's state space canonical decomposition scheme is given and some structures of the associated lattice diagram are discussed.

Some additional lattices with special structures that contain the lattices of invariant subspaces as their sublattices are introduced. Using the properties of these lattices, generalized controllability and observability criteria are developed and described.

Finally, application of the properties of these lattices to perfect output controllability, functional output controllability, unknown-input state observability, functional input observability, and feedback compensation problems related to the dynamical system of interest are demonstrated.

Because of the fact that the "replacement" of a finite-dimensional linear vector space by a certain lattice constitutes the basis for our new approach, it is conceivable that most of the theory of linear time-invariant dynamical systems originally developed on finite-dimensional linear vector spaces can be analogously treated and possibly generalized and extended "on lattices" in a natural manner. This is indeed the case and the research work reported in this thesis especially the broad brush treatment of many different concepts pulled together from various areas of linear dynamical systems theory in Chapter V bears witness to this fact. However, due to the fact that our lattice-theoretic approach is at an embryonic stage of development, the question as to how this theory will contribute to the investigation of various analysis and synthesis problems in linear dynamical systems remains to be answered only by large scale research in this area.

In addition to our claim that almost any aspect of linear time-invariant dynamical systems can be studied using our lattice-theoretic approach, we will explicitly specify some problems for further investigation. These problems were uncovered during the course of our research and can be investigated as natural extensions of some of the concepts developed in this thesis.

1. The structures and properties of lattices of linear subspaces of an infinite-dimensional linear vector space can be investigated and lattice-theoretic characterizations of the properties of linear dynamical systems with infinite-dimensional state and output spaces developed. This is undoubtedly a challenging problem since it seems that very little has been done on lattices of linear subspaces of an infinite-dimensional linear vector space and, on the other hand, the state space theory of infinite-dimensional linear dynamical systems is definitely lacking adequate structure.

2. Because of the inherent relationships among the structures of matrices of linear operators, their characteristic polynomials, their eigenvalues, and certain related linear spaces which have been extensively used in the study of linear dynamical systems, it is possible to

develop various structures of the lattice of linear subspaces of a finite dimensional linear vector space using arguments essentially based on the properties of polynomials. This line of reasoning then can be used to investigate various problems in different areas of linear dynamical systems. Since the theory of polynomials and related areas are fully developed, this approach seems very promising.

3. Using the lattice-theoretic approach one can investigate some properties of composite linear time-invariant dynamical systems. Controllability and observability of these systems which are formed by the interconnection of multivariable systems were first introduced by Gilbert [29] in 1963. Since then other authors [11,22,49] have further investigated various properties of both parallel-connected and tandem-connected composite systems using conventional transfer function and vector space techniques. It seems that our approach can be applied to the study of this particular class of systems in a fairly straightforward manner.

4. In Chapter IV two important special-structure lattices $K(B)$ and $\bar{K}(B)$ both of which contain the lattice of invariant subspaces $L_A^*(V)$ as a sublattice were developed and their properties characterized. For the lattice $L_A^*(V)$ Theorem 2.6.2 provides an algorithmic generation scheme. Investigating the possibility of developing similar schemes for the lattices $K(B)$ and $\bar{K}(B)$ might lead to some important theoretical and computational results.

5. In Chapter V we formulated a number of important concepts. Especially the noninteraction control problem and its related offshoots

seem to lend themselves well to a lattice-theoretic analysis. Because of their great significance in the analysis and design of modern control systems, this particular class of problems warrants a good deal of research.

APPENDIX

Some Concepts from Lattice Theory

For the purpose of easy reference, we have collected here the definitions of some mathematical terms and some basic theorems from lattice theory which appear in this thesis. For comprehensive treatments of lattice theory references [1,26,31] and [65] may be consulted.

Definition 1. A *poset* (partially ordered set), $<P,\subseteq>$ is a set P with a binary relation $\subseteq$ such that for all $x,y,z \in P$ the following conditions hold.

P1. $x \subseteq x$ (Reflexivity)

P2. $x \subseteq y,\ \ y \subseteq x \Rightarrow x = y$ (Antisymmetry)

P3. $x \subseteq y,\ \ y \subseteq z \Rightarrow x \subseteq z$ (Transitivity)

If $x \subseteq y$ and $x \neq y$, we write $x \subset y$, and say that x is "properly contained" in y. It is emphasized that $\subseteq$ is not necessarily set inclusion. $<P,\subseteq>$ is simply any set with any relation that is reflexive, antisymmetric and transitive.

Definition 2. Let $<P,\subseteq>$ be a poset of elements $x,y,\ldots$; if $x \subset y$ and there is no element $m \in P$ such that $x \subset m \subset y$ we say that y *covers* x and denote this relation by $x \lessdot y$.

Definition 3. A *chain* is a poset in which for each pair of distinct elements x,y we have either $x \subset y$ or $y \subset x$. We note that any subset of a chain is itself a chain and is called a *subchain*.

Definition 4. By a *least element* of a subset S of a poset P, we mean an element $x \in S$ such that $x \subseteq m$ for all $m \in S$. By a *greatest element* of S, we mean an element $y \in S$ such that $y \supseteq m$ for all $m \in S$. The unique least and greatest elements of the whole poset P, when they exist, are called the *universal bounds* or *zero* and *unit elements* of P, and are denoted by 0 and I, respectively.

Thus a poset P has universal bounds 0 and I, iff $0 \subseteq m \subseteq I$ for all $m \in P$.

The concepts of "least" and "greatest" elements are different from the concepts of "minimal" and "maximal" elements. A *minimal element* of a subset S of a poset P is an element x such that $m \subset x$ for no $x \in S$; a *maximal element* is defined dually. Clearly, a least element must be minimal and a greatest element maximal, but the converse is not true. Whereas S can have many minimal elements, it can have only one least element. For a chain, the notions of minimal and least (maximal and greatest) are equivalent.

The converse of a binary relation ρ is, by definition, the relation $\check{\rho}$ such that $x\check{\rho}y$ (read, "x is in the relation $\check{\rho}$ to y") if and only if $y\rho x$. Thus the converse of the relation "includes" is the relation "is included in." From inspection of P1-P3 of Definition 1 the following theorem is obvious.

Theorem 1 (Duality Principle). The converse of any partial ordering is itself a partial ordering.

Definition 5. The *dual* of a poset P is that poset P defined by the converse partial ordering relation on the same elements.

Definition 6. By the *length* of a finite chain C consisting of n elements, i.e., being of the form $x_1 \subset x_2 \subset \ldots \subset x_n$ is meant the non-negative integer n, and the length of a chain consisting of an infinite number of elements is symbolized by ∞. Then, the length $\ell[P]$ of the poset P is the length of the supremum of the lengths of all subchains in P.

Definition 7. In a poset P of finite length with a null element 0, the *height* or "dimension" $h[x]$ of an element $x \in P$ is the supremum of the lengths of the chains $0 = x_0 \subset x_1 \subset \ldots \subset x_k = x$ between 0 and x. If P has a unit element I, then clearly $h[I] = \ell[P]$.

Definition 8. The *product* $P \times Q$ of any two posets P and Q is the set of all pairs (x,y) with $x \in P$, $y \in Q$, partially ordered by the rule that $(x_1,y_1) \subseteq (x_2,y_2)$ if and only if $x_1 \subseteq x_2$ in P and $y_1 \subseteq y_2$ in Q.

Definition 9. If $<L,\subseteq>$ is a poset such that any two elements $x,y \in L$ have an infimum or "meet," $x \cap y$, then $<L,\subseteq>$ is called a *meet semi-lattice*. Dually, if any two elements $x,y \in L$ have a supremum or "join," $x \cup y$, then $<L,\subseteq>$ is called a *join semi-lattice*. If there exist both $x \cap y$ and $x \cup y$ in L, then L is called a *lattice*.

Theorem 2. $<L,\cup,\cap>$ is a lattice iff for all $x,y,z \in L$ the following conditions hold:

L1. $x \cap x = x$, $x \cup x = x$ (Idempotency)

L2. $x \cap y = y \cap x$, $x \cup y = y \cup x$ (Commutativity)

L3. $x \cap (y \cup z) = (x \cap y) \cap z$, $x \cup (y \cup z) = (x \cup y) \cup z$ (Associativity)

L4. $x \cap (x \cup y) = x \cup (x \cap y) = x$ (Absorption)

Moreover, $x \subseteq y$ is equivalent to each of the following conditions:

$x \cap y = x$ and $x \cup y = y$ (Consistency)

Definition 10. A nonempty subset S of elements of a lattice L which contains the meet and join of any two of its elements, i.e., $x,y \in S$ implies that $x \cap y \in S$ and $x \cup y \in S$, is called a *sublattice* of L.

Definition 11. A lattice in which every subset has an infinum and a supremum is called a *complete lattice*. It is clear that any finite lattice or any lattice of finite length is complete.

Definition 12. A lattice L is *modular* if it satisfies the following modular identity:

$$x \subseteq z \text{ implies } x \cup (y \cap z) = (x \cup y) \cap z, \quad \text{for all } x,y,z \in L$$

Theorem 3. A sublattice of a modular lattice is modular.

Definition 13. A lattice L is *distributive* if it satisfies both of the following distributive identities:

$$x \cap (y \cup z) = (x \cap y) \cup (x \cap z),$$

$$x \cup (y \cap z) = (x \cup y) \cap (x \cup z), \quad \text{for all } x,y,z \in L.$$

Theorem 4. Every sublattice of a distributive lattice is distributive.

Theorem 5. Any chain is a distributive lattice.

Theorem 6. The direct product $L_1 \times L_2 \times \ldots \times L_n$ of the lattices $L_1,L_2,\ldots,L_n$ is a distributive lattice iff each L_i, $i = 1,2,\ldots,n$ is distributive.

Definition 14. For any pair of comparable elements m,n of a lattice L where $m \subseteq n$, the subset S of L consisting of all elements $x, m \subseteq x \subseteq n$, is called the *interval* [m,n].

Definition 15. Let x be an element of the interval [m,n] of a lattice L; if an element y exists in L such that $x \cap y = m$, $x \cup y = n$, then y is called a *complement* of x relative to the interval [m,n]. We note that such a y must belong to the interval, because $m = x \cap y \subseteq x \cup y = n$; and that the relation is symmetric, x being a complement of y relative to the interval.

Definition 16. If every element x of an interval [m,n] has one complement relative to [m,n], the interval is said to be *complemented*. If every interval in a lattice is complemented, the lattice is said to be *relatively complemented*.

Definition 17. Let x be an element of a lattice L with zero element 0 and unit element I. If there exists in L an element y such that $x \cap y = 0$ and $x \cup y = I$, then y is called a *complement* of x.

Definition 18. If an element x of a lattice L with 0 and I has at least one complement, the lattice is said to be *complemented.* It follows that a relatively complemented lattice with 0 and I is complemented; but a complemented lattice need not be relatively complemented.

Definition 19. A complemented distributive lattice is called a *Boolean algebra.*

Definition 20. An element x of a lattice L is said to be *meet-reducible* if elements m and n can be found in L such that $x = m \cap n$ with $m \supset x$, $n \supset x$. If such elements cannot be found, x is said to be *meet-irreducible.* If elements p and q exist in L such that $x = p \cup q$ with $p \subset x$, $q \subset x$, then x is said to be *join-reducible,* otherwise *join-irreducible.*

Definition 21. If in a lattice L, $x = \bigcap_{i=1}^{n} x_i$, $x, x_i \in L$, $i = 1, 2, \ldots, n$, where x_i are all meet-irreducible and $x = \bigcap_{i=2}^{n} x_i$, then x_1 is said to be *redundant* in the decomposition $x = \bigcap_{i=1}^{n} x_i$. If no x_i is redundant, the decomposition is said to be *without redundancy.*

Dually, if in a lattice L, $x = \bigcup_{i=1}^{n} y_i$, $x, y_i \in L$, $i = 1, 2, \ldots, n$, where y_i are all join-irreducible and $x = \bigcup_{i=2}^{n} y_i$, y_1 is said to be redundant in the decomposition $x = \bigcup_{i=1}^{n} y_i$. If no y_i is redundant, the decomposition is said to be without redundancy.

<u>Definition 22</u>. A non-empty subset X of a lattice L is called an *ideal* of L iff the following conditions are satisfied:

1. $x,y \in X$ implies $x \cup y \in X$.
2. $x \in X$, $z \in L$ imply $x \cap z \in X$.

<u>Definition 23</u>. A non-empty subset Y of a lattice L is called a *dual ideal* or *filter* of L iff the following conditions hold:

1. $x,y \in Y$ implies $x \cap y \in Y$.
2. $x \in Y$, $z \in L$ imply $x \cup z \in Y$.

<u>Definition 24</u>. Let m be an arbitrary element of a lattice L. Then the subsets $X = \{x \in L: x \subseteq m\}$ and $Y = \{y \in L: y \supseteq m\}$ are called the *principal ideal* and the *dual principal ideal* of L, respectively, generated by m. X and Y are denoted by m^{∇} and m^{Δ}, respectively.

<u>Theorem 7</u>. In a finite lattice every ideal is principal.

<u>Theorem 8</u>. In a lattice of finite length every ideal is principal.

<u>Definition 25</u>. Let L_1 and L_2 be lattices and σ a mapping of L_1 into L_2. If for all $x,y \in L_1$ we have in L_2 $\sigma(x \cap y) = \sigma(x) \cap \sigma(y)$, then σ is said to *preserve meets* and is called a *meet-homomorphism*. If for all $x,y \in L_1$ we have in L_2, $\sigma(x \cup y) = \sigma(x) \cup \sigma(y)$, then is said to *preserve joins* and is called a *join-homomorphism*. If σ preserves both meets and joins, it is called a *lattice homomorphism*. If L_2 is the same lattice as L_1, the homomorphism is called an *endomorphism*.

Theorem 9. The homomorphic image H of a lattice L_1 under $\sigma: L_1 \to L_2$ is a sublattice of L_2.

Definition 26. If a lattice homomorphism $\sigma: L_1 \to L_2$ is both one-to-one and onto, then it is called an *isomorphism*. If L_2 is the same lattice as L_1, σ is called an *automorphism*.

Definition 27. Let P_1, P_2 be posets and let σ be a mapping of P_1 into P_2 such that if $x \subseteq y$ in P_1, then $\sigma(x) \subseteq \sigma(y)$ in P_2; σ is said to *preserve order*.

Theorem 10. Any lattice homomorphism preserves order.

Definition 28. A lattice L with 0 and I is called *orthocomplemented* when there is a mapping $x \mapsto x^{\perp}$ of L into itself satisfying the following three conditions.

1. $x^{\perp}$ is a complement of x.
2. $x \subseteq y$ implies $x^{\perp} \supseteq y^{\perp}$.
3. $x^{\perp\perp} = x$ for every x.

We call $x^{\perp}$ the orthocomplement of x. When $x \subseteq y^{\perp}$ we say that x and y are orthogonal and write $x \perp y$.

By conditions 2 and 3 of Definition 28, the orthocomplementation $x \mapsto x^{\perp}$ is a dual isomorphism of L onto itself. Hence L and its dual L^* are isomorphic by the orthocomplementation. From this fact we have

$$0^{\perp} = I, \quad I^{\perp} = 0, \quad (x \cup y)^{\perp} = x^{\perp} \cap y^{\perp}, \quad \text{and} \quad (x \cap y)^{\perp} = x^{\perp} \cup y^{\perp}.$$

Condition 1 of Definition 28 may be replaced by $x \cap x^{\perp} = 0$, since this implies

$$x \cup x^{\perp} = (x^{\perp} \cap x)^{\perp} = 0^{\perp} = I.$$

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